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Paul SORBA

ALGEBRES DE LIE ET PARTICULES ELEMENTAIRES :

1) Relation entre l'algèbre des charges et le groupe de classification des hadrons
2) Sous-groupes connexes de Poincare et champs électromagnétiques
3) Les algèbres de Schrödinger, Galilée et conforme en Mécanique Quantique

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A MES PARENTS,

A DENISE, GERALDINE et ISABELLE

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"Plus on regarde, plus on voit.
Mais plus aussi on voit où il
faut regarder".

P. TEILHARD DE CHARDIN
(Le phénomène humain)

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INTRODUCTION GENERALE

Un des aspects les plus importants de la "révolution contemporaine" concernant la conception des particules élémentaires se trouve sans doute dans la correspondance entre les lois de conservation et de sélection parmi les interactions des particules et les symétries de base inhérentes à la structure de l'univers. Ces propriétés de symétrie et d'invariance sont fréquemment géométriques dans leur origine. L'ingrédient mathématique parfaitement adapté dans de nombreux cas est alors fourni par la théorie des groupes et en particulier par la théorie des groupes de Lie.

Considérés en tant que groupes d'invariance et de symétrie, leur utilisation ne s'arrête pas à ce champ d'action déjà très vaste. Signalons à ce propos les études concernant les groupes de non invariance ("spectrum generating algebras") avant de rappeler un des points les plus remarquables dans les récents développements de cette science, à savoir la correspondance entre les états des particules observées et les représentations de certains groupes de Lie. Ces relations ont permis de relier et aussi de prédire les propriétés de beaucoup de particules connues, ainsi que l'existence de particules découvertes plus recemment (la particule Ω par exemple). Enfin grâce à ces relations, des connections entre certains aspects dynamiques des interactions entre particules ont pu être déduites.

C'est à l'étude de certains de ces problèmes que cette thèse est consacrée. Trois parties que nous allons rapidement présenter la composent.

La première partie est une étude de la relation entre l'algèbre des courants et le groupe de classification des hadrons.

La transformation unitaire reliant l'algèbre SU(6) des charges aux générateurs de classification, proposée sur des considérations algébriques et phénoménologiques par Buccella et al. en 1970, puis calculée explicitement dans le modèle du champ de quarks libres par Melosh en 1973, est retrouvée ici en faisant le produit de représentations du groupe de Poincaré pour un système de quarks. Cette transformation apparaît alors comme une rotation de Wigner du spin des quarks. L'approche groupale utilisée paraît exprimer plus clairement le concept de quark courant et quark constituant, et permet de préciser certains points concernant le groupe de classification.

Cette transformation est appliquée dans le cas des réactions $K^- p \to \pi^0 \Lambda(1405)$ et $K^- p \to \pi^0 \Lambda(1520)$ à des calculs de contributions, dans la partie imaginaire de l'amplitude de diffusion, de résonances intermédiaires Σ^*, ainsi qu'à la détemination de largeurs de désintégration $\Gamma_{\Sigma^* \to K^- p}$ et $\Gamma_{\Sigma^* \to \pi^0 \Lambda}$.

Si les symétries internes jouent un rôle de premier ordre dans la première partie, ce sont les symétries externes qui sont à la base des des travaux constituant les seconde et troisième parties.

La seconde partie est plus exactement consacrée à des problèmes de cinématique relativiste. Elle comprend d'abord les classifications (modulo une conjugaison, puis modulo un isomorphisme) de tous les sous-groupes connexes du groupe de Poincaré. Cette classification est suivie, comme application, de la détermination et de la classification des champs électromagnétiques admettant comme groupe de symétrie un sous-groupe de Poincaré. L'intérêt de cette dernière étude est en particulier dans la possibilité de déduire, pour des particules soumises à l'action d'un champ électromagnétique extérieur invariant sous un sous-groupe de Poincaré, des propriétés concernant leur mouvement, à partir de l'étude du sous-groupe d'invariance.

Enfin, notre intérêt est porté vers les basses énergies dans la troisième partie, où les groupes de Galilée et de Schrödinger occupent une place particulière.

C'est en effet la structure de l'algèbre de Schrödinger qui suggèra le développement d'une méthode permettant d'agrandir une algèbre d'invariance (dépendant explicitement du temps) d'un système quantique dont une réalisation est connue en termes de certaines variables en une algèbre d'invariance plus grande en termes des mêmes variables. Des applications, intéressant la physique nucléaire, concernant des systèmes de n particules en interaction ont pu être effectuées. De plus nous donnons une classification de certains potentiels d'interaction par rapport à leurs groupes d'invariance respectifs. Ajoutons que, afin de rendre plus aisément utilisable la méthode précitée, laquelle fait intervenir les dérivations de l'algèbre d'invariance considérée, nous établissons un théorème permettant de calculer assez simplement les automorphismes d'une algèbre de Lie.

D'autre part nous proposons une nouvelle base de l'algèbre conforme, laquelle fait apparaître deux algèbres de Schrödinger (agissant sur un espace temps à 2+1 dimensions) conjuguées. Cette structure non relativiste de l'algèbre conforme permet de discuter certains modèles de hadrons basés sur des particules sans masse collinéaires.

Enfin, nous présentons les principaux résultats concernant la classification, établie par ailleurs, des sous-groupes connexes de Galilée et de son extension centrale.

P A R T I E I

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RELATION ENTRE L'ALGEBRE DES CHARGES ET LE GROUPE DE CLASSIFICATION DES HADRONS

0 - INTRODUCTION

Si l'on considère l'attitude des physiciens des particules face à la prolifération des résonances hadroniques, on s'aperçoit que le support mathématique de la physique des hadrons ces dix dernières années se limite au groupe SU(6) -ou plutôt à son algèbre de Lie (et à certains de ses sous-groupes : SU(2) , SU(3) , SU(3) x SU(3)).

En fait deux SU(6) physiquement différents coexistent

- SU(6) : groupe de classification des hadrons $SU(6)_W^{class}$, que l'on remplace volontiers par U(6) x U(6).
- SU(6) : groupe dont une réalisation de son algèbre de Lie est l'algèbre des "good" charges dans la théorie de l'algèbre des courants : $SU(6)_W^{curr}$.

Il a été montré que algèbre de classification et algèbre des charges peuvent être reliées par une transformation unitaire.

Les premiers à découvrir cette transformation furent Buccella, Kleinert, Savoy, Celeghini et Sorace [1]. Construite à partir de considérations algébriques et phénoménologiques, elle fut testée avec succès dans de nombreux calculs.

Récemment, Melosh [2] (Caltech Thèse 1973) obtint cette transformation par un traitement mathématique plus rigoureux en étudiant le champ de quarks libres. Il montra que cette transformation est en fait une transformation de Foldy-Wouthuysen, comme le prédisait Gell-Mann quelques années auparavant.

Cette transformation peut être retrouvée à partir de l'étude du groupe de Poincaré P pour un système de quarks.[3] En fait, on peut voir que l'opérateur de mixing apparaît comme une rotation de Wigner dans les coefficients de Clebsch-Gordon provenant de la décomposition en représentations irréductibles de P .

C'est cette étude qui constituera l'objet du chapitre II. Le traitement "groupal" considéré nous paraît exprimer plus clairement comment sont reliés quarks courants et quarks constituants -et par suite algèbre des charges et algèbre de classification- ainsi que certains points concernant le groupe de classification.

Auparavant sont définies et comparées dans le premier chapitre, algèbre de charge et algèbre de classification.

Une discussion des résultats du Chapitre II, ainsi qu'une étude rapide des états à 3 particules (cas des baryons) forment le Chapitre III.

Enfin, dans le Chapitre IV, la transformation précitée est appliquée dans le cadre des réactions $K^- p \to \pi^0 \Lambda(1405)$ et $K^- p \to \pi^0 \Lambda(1520)$ aux calculs des contributions, dans la partie imaginaire de l'amplitude de diffusion, des résonances Σ^*, ainsi qu'à la détermination des largeurs de désintégration $\Gamma_{K^- p \to \Sigma^*}$ et $\Gamma_{\Sigma^* \to \pi^0 \Lambda}$.

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CHAPITRE I : ALGEBRE DE CHARGE ET ALGEBRE DE CLASSIFICATION DES HADRONS

A) SU(6) et U(6) x U(6) x O(3) DE CLASSIFICATION ; LES QUARKS

$SU(6)_s$ est né, de même que son sous-groupe SU(3), comme symétrie brisée.

1. Modèle des quarks (q).

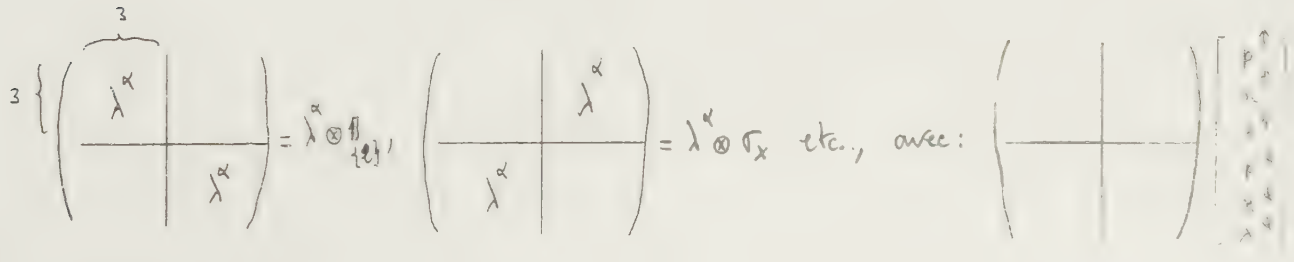
Se rappelant les bases de la symétrie unitaire, on imagine 3 particules q de spin $\frac{1}{2}$ (p, n, λ) formant une base de l'espace de la représentation $\{3\}$ de SU(3). Chacune d'elles pouvant exister sous 2 états de spin $\uparrow \downarrow$, ces 2 états forment une base de la représentation $\{2\}$ du groupe de spin SU(2). Les 6 états ainsi obtenus forment une base de $V_3 \otimes V_2 = V_6$ espace de dimension 6 sur lequel agit irréductiblement SU(3) x SU(2).

Ce sous-groupe est maximal dans SU(6), groupe des opérateurs unitaires, unimodulaires sur V_6 .

Les 35 générateurs de SU(6) s'écrivent

$$\underbrace{\lambda^\alpha \otimes 1_{(2)}}_{SU(3)} \quad ; \quad \underbrace{1_{(3)} \otimes \sigma_i}_{SU(2)} \quad ; \quad \lambda^\alpha \otimes \sigma_i \qquad (\alpha = 1,2,\dots 8 ; i = 1,2,3)$$

Sous forme de matrices 6x6 , on aura :



Par rapport à SU(3) x SU(2), la représentation fondamentale $\{6\}$ se réduit à $(3,2)$.
On associera aux antiquarks $(\bar{q})$ la représentation $\{\bar{6}\}$, se réduisant à $(\bar{3},2)$.

2. Classification des hadrons.

Il apparaît que les états de hadrons au repos sont bien décrits par un modèle de quarks : l'hypothèse est que les hadrons "low-lying" agissent comme s'ils étaient formés d'une paire $q\,\bar{q}$ dans le cas des mésons, ou de 3 quarks $q\,q\,q$ dans le cas des baryons.

Notons que des configurations plus complexes telles que $q\,\bar{q}\,q\,\bar{q}$ ou $q\,q\,q\,q\,\bar{q}$ sont appelées usuellement exotiques ; elles peuvent apparaître dans des états multiparticulaires, mais il peut y avoir aussi des résonances leur correspondant.

On peut dire que c'est en 1964 que Gursey et Radicati et Sakita [1] proposèrent que SU(6) était un groupe de symétrie approximative pour les interactions fortes dans la limite non relativiste. En fait, ils étendaient aux particules élémentaires l'idée du groupe de symétrie SU(4) proposé par Wigner pour les nucléons.

Décomposant en représentations irréductibles les produits de représentations de SU(6):

$$\{6\} \times \{\bar{6}\} = \{1\} + \{35\}$$

$$\{6\} \times \{6\} \times \{6\} = \{56\} + \{70\} + \{70\} + \{20\}$$

on remarque ainsi qu'un baryon est classé dans une -ou plusieurs- des représentations $\{56\}$, $\{70\}$ ou $\{20\}$, tandis qu'un méson sera classé dans les représentations $\{1\}$ et $\{35\}$.

Il apparaît plus pédagogique de considérer le groupe $SU(6)_q \times SU(6)_{\bar{q}}$ avec $q \in (6,1)$ $\bar{q} \in (1,\bar{6})$; baryons et mésons seront alors classés respectivement dans les représentations $(56,1)$, $(70,1)$, $(20,1)$ et $(6,\bar{6})$ de $SU(6) \times SU(6)$.

Enfin, comme le spin des résonances hadroniques connues peut être assez élevé, un moment angulaire orbital $\vec{L}$ est introduit et le spin du hadron est défini par $\vec{J} = \vec{L}+\vec{S}$. En d'autres termes, on est amené à considérer le groupe $SU(6)_q \times SU(6)_{\bar{q}} \times O(3)_L$. On aura ainsi, par exemple, des baryons classés dans la $56, L = 0$ ou la $70, L = 1$ (voir Appendice).

3. $SU(6)_W$

Les défauts du groupe décrit ci-dessus, et noté $SU(6)_s \times O(3)$, comme symétrie brisée ont été longuement étudiés [2] . Il est statique, c'est-à-dire utilisable seulement pour des particules au repos. Il est par suite complétement inintéressant pour des processus collinéaires de désintégration, comme

$\rho \rightarrow \pi\ \gamma$ ou $\Delta \rightarrow N\ \pi$

mésons mésons — baryon baryon mésons

$S=1$ $S=0$ — $S = 3/2 \notin S=\frac{1}{2} \otimes S=0$.

Un autre sous-groupe de $SU(6) \times SU(6) \times O(3)$ fut alors proposé : $SU(6)_W \times O(3)$, lequel remédie à la précédente difficulté. Ce $SU(6)_W$ fut découvert par Lipkin et Meshkow [3] cherchant à expliquer l'indépendance de spin dans les interactions fortes (approximativement) en postulant que l'hamiltonien des interactions fortes H_{st} commute avec un ensemble de 35 opérateurs W_i^α formant une algèbre $SU(6)$.

Pratiquement, ce $SU(6)_W$ s'obtient en remplaçant la partie $SU(2)_s$ de spin S engendrée par les matrices $\{\sigma_x, \sigma_y, \sigma_z\}$ par $\{\gamma^0\hat{\sigma}_x, \gamma^0\hat{\sigma}_y, \hat{\sigma}_z\}$

avec $\hat{\sigma}_i = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$ matrice 4x4, et $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

On vérifie que ce nouvel $SU(2)$ -noté $SU(2)_W$ - commute avec les boosts de Lorentz dans la direction z (générateurs des boosts $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$).

On formera l'algèbre $SU(6)_W$ en combinant les générateurs λ^α de $SU(3)$ à ceux de $SU(2)_W$ de la même façon qu'avaient été combinés les λ^α aux générateurs de $SU(2)_s$.

$SU(6)_W$ est un sous-groupe de $SU(6)_q \times SU(6)_q$: une base est en effet fournie par les générateurs :

$$\left(X_W\right)_\mu^\alpha = \left(X_\mu^\alpha\right)_q \times I_{\bar{q}} \pm I_q \times \left(X_\mu^\alpha\right)_{\bar{q}} \qquad \begin{matrix} \text{avec } + \text{ pour } \mu = 0, 3 \\ \text{avec } - \text{ pour } \mu = 1, 2. \end{matrix}$$

Ainsi $SU(6)_W$ commute avec les boosts dans la direction z

L'intérêt de l'introduction de ce "spin W" est donc qu'une particule se déplaçant avec un moment arbitraire dans la direction z a le même spin W que la particule correspondante au repos. Par suite $SU(6)_W$ apparaît comme le sous-groupe du groupe de symétrie statique $SU(6)_q \times SU(6)_{\bar{q}}$ approprié pour décrire les particules se déplaçant suivant l'axe des z, et peut-être, les désintégrations à 2 corps et les processus collinéaires dans un cadre relativiste.

Remarquons que pour les quarks $\vec{W} = \vec{S}$, tandis que pour les antiquarks $W_{x,y} = -S_{x,y}$, $W_z = S_z$. Une simple utilisation des coefficients de Clebsch-Gordan permet de vérifier que les réactions signalées ci-dessus sont permises :

$$\rho \longrightarrow \pi\,\pi \qquad ; \qquad \Delta \longrightarrow N\,\pi$$
$$W=0 \qquad W=1 \otimes W=1 \qquad\qquad W=3/2 \qquad W=\tfrac{1}{2} \otimes W=1$$

La plus fameuse prédiction de $SU(6)_W$ est la relation de Johnson-Treiman [4]

$$\sigma_T(\pi^-) - \sigma_T(\pi^+) = \sigma_T(\bar{K}^0) - \sigma_T(K^0) = \frac{1}{2}\left[\sigma_T(K^-) - \sigma_T(K^+)\right]$$

bien vérifiée expérimentalement.

B) SU(6) ET ALGEBRE DES COURANTS [5]

1. Algèbre chirale :

Rappelons que les interactions électromagnétiques et faibles sont bien décrites par des courants.

La théorie V-A postule des termes vectoriels et axiaux dans la partie hadronique des courants faibles, en parfaite analogie avec la partie leptonique.

Appelons charges vectorielles $Q^\alpha(t) = \int d^3x\, j_0^\alpha(x)$ $\alpha \begin{cases} 1 \\ 2 \end{cases} \Delta S=0$

et charges axiales $Q_5^\alpha(t) = \int d^3x\, j_{05}^\alpha(x)$ $\alpha \begin{cases} 4 \\ 5 \end{cases} \Delta S=1$

Ajoutons la charge électromagnétique (Gell-Mann - Nishijima)

$$Q^{EM} = I^3 + \tfrac{1}{2} Y = Q^3 + \frac{1}{\sqrt{3}} Q^8$$

Alors l'hypothèse CVC postule que Q^1, Q^2, Q^3 sont les générateurs du groupe d'isospin SU(2) tandis que Y est l'hypercharge : ces opérateurs sont alors conservés. (rappel : CVC = courant vecteur conservé).

Afin d'incorporer le principe de l'universalité des interactions faibles entre hadrons et leptons, cette algèbre SU(2) x U(1) est étendue à $Q^4(t)$ et $Q^5(t)$ en supposant que toutes les charges vectorielles mesurables forment une algèbre SU(3)

$$\left[Q^\alpha(t), Q^\beta(t)\right] = i f^{\alpha\beta\gamma} Q^\gamma(t) \qquad (\alpha,\beta,\gamma = 1,2,\ldots 8)$$

Ces 8 charges sont identifiées avec les générateurs du SU(3) de symétrie, notamment à l'aide du théorème d'Ademollo-Gatto [6].

Finallement, Gell-Mann proposa l'hypothèse qu'une algèbre SU(3) x SU(3) est obtenues en ajoutant aux charges vectorielles les charges axiales :

Algèbre chirale

$$\begin{cases} \left[Q^\alpha(t), Q^\beta(t)\right] = i f^{\alpha\beta\gamma} Q^\gamma(t) \\ \left[Q^\alpha(t), Q_5^\beta(t)\right] = i f^{\alpha\beta\gamma} Q_5^\gamma(t) \\ \left[Q_5^\alpha(t), Q_5^\beta(t)\right] = i f^{\alpha\beta\gamma} Q^\gamma(t) \end{cases}$$

(la structure SU(3) x SU(3) apparaissant en définissant les nouveaux opérateurs $Q_\pm^\alpha(t) = \frac{1}{\sqrt{2}}\left\{Q^\alpha(t) \pm Q_5^\alpha(t)\right\}$: ils vérifient en effet :

$$\left[Q_\pm^\alpha(t), Q_\pm^\beta(t)\right] = i f^{\alpha\beta\gamma} Q_\pm^\gamma(t)$$

$$\left[Q_+^\alpha(t), Q_-^\beta(t)\right] = 0$$

2. Saturation de l'algèbre chirale et référentiel $p_z \to \infty$

La réalisation de l'algèbre SU(3) x SU(3) par des états hadroniques est appelée saturation. C'est grâce à elle que de nombreuses informations sur les mésons et les baryons sont obtenues. Plus précisément, nous serons conduits à un ensemble de règles de somme ; par exemple:

$$\langle a,\vec{p} \,|\, \left[Q_5^\alpha(t), Q_5^\beta(t)\right] \,|\, a,\vec{p}\rangle = i f^{\alpha\beta\gamma} \langle a,\vec{p} \,|\, Q^\gamma(t) \,|\, a,\vec{p}\rangle$$

$$= \sum_n \left\{ \langle a,\vec{p} \,|\, Q_5^\alpha(t) \,|\, n,\vec{p}\rangle \langle n,\vec{p} \,|\, Q_5^\beta(t) \,|\, a,\vec{p}\rangle - (\alpha \leftrightarrow \beta) \right\}$$

entre états à 1 particule identiques, $\{n\}$ étant un ensemble complet d'états hadroniques.

Il apparaît alors très intéressant de se placer dans le "référentiel $p_z \to \infty$ ". Résumons en bref les principaux avantages :

i) Les éléments de matrice $\langle a, \infty | Q_5^\alpha(t) | n, \infty \rangle$ sont alors pris à $q^2 = (p_a - p_n)^2 = \left(\sqrt{m_a^2 + \vec{p}^2} - \sqrt{m_n^2 + \vec{p}^2}\right)^2 = 0$, donc ne dépendent plus du repère choisi, ni de $\vec{p}$.

ii) Les états à plusieurs particules (plus précisément les états avec $m_n^2 \to \infty$) ne contribuent plus à la règle de somme.

iii) La condition $q^2 = 0$ permet d'appliquer l'hypothèse PCAC (courant axial conservé partiellement) pour relier les éléments de matrice de charge axiale aux couplages pioniques.

3. "Bonnes" et "mauvaises" charges

Il est possible de définir d'autres charges comme intégrales d'espace de courants ; ainsi a pu être déterminée une algèbre U(6,6) de charges (144 générateurs) contenant comme sous-algèbre l'algèbre chirale SU(3) x SU(3).

La moitié de ces charges sont déclarées "mauvaises" (bad) car car dans la limite $p_z \to \infty$ ou bien elles donnent 0 de chaque côté du signe égal dans les règles de somme correspondantes, ou bien les règles de somme mettant en jeu ces charges ne sont pas définies. Les autres charges fournissent des éléments de matrice non nuls quand $p_z \to \infty$; elles sont appelées "bonnes" (good). En fait elles se trouvent identiques par paires dans cette limite. On exhibe ainsi une algèbre SU(6) notée $SU(6)_W^{curr}$ = l'algèbre des bonnes charges. Notons que cette algèbre contient l'algèbre chirale SU(3) x SU(3), et des intégrales de courants tensoriels.

Signalons enfin que l'hypothèse PCAC utilisée avec $q^2 = 0$ a conduit en particulier à la relation d'Adler-Weisberger [7] :

$$\left(\frac{G_A}{G_V}\right)^2 = 1 + \frac{2}{\pi} \frac{M}{g_{\pi n}} \int_{\nu_L}^{\infty} \frac{q \, d\nu_L}{\nu_L^2} \left[\sigma_T^{\pi^- p}(\nu_L) - \sigma_T^{\pi^+ p}(\nu_L) \right]$$

$\frac{G_A}{G_V}$ = constante de renormalisation de la désintégration β .

4. Charges du genre lumière :

Les avantages du "référentiel $p_z \to \infty$ " dans l'étude des éléments de matrice de charge -mentionnés ci dessus- peuvent être préservés, sans qu'il soit nécessaire de se restreindre aux états à moment infini, en considérant les charges du genre lumière [8]

Réalisation des charges à t égaux ; $p_z \to \infty$ $\Longleftrightarrow$ Réalisation des charges L.L (light-like) $\forall \vec{p}$

Il apparaît en effet que si l'on prend le commutateur de 2 charges :

$$Q^{i'}(n) = \int d^4x \; n^a . J_a^{i'}(x) \; \delta(n.x)$$

avec n du genre temps, et que l'on considère dans la règle de somme correspondante n^2 comme variable, alors la limite $n^2 \to 0_+$ est équivalente à la limite usuelle $p_z \to \infty$.
Ces opérateurs, considérés à $n^2 = 0$, sont donc des intégrales de courant sur l'hyperplan lumière : $n.x = 0$, $n^2 = 0$, d'où le nom de charges du genre lumière (par opposition aux charges du genre espace). Les propriétés de ces charges présentent un grand intérêt pour l'étude qui va suivre : voir Chapitre II .

C) RELATION ENTRE LES 2 ALGEBRES SU(6)

1. Comparaison des 2 algèbres SU(6).

Dans la description rapide précédente, on a vu que $SU(6)_W^{class}$ et $SU(6)_W^{curr.}$ contenait un même sous-groupe SU(3).
Mais l'identification ne peut aller plus loin. De nombreux faits expérimentaux empêchent de confondre les 2 SU(6) : une telle confusion entrainerait par exemple* : $\frac{G_A}{G_V} = 1,66$ au lieu de la valeur bien connue $\frac{G_A}{G_V} = 1,23$.

2. Quarks constituants - Quarks courants

Par suite des actions différentes des 2 SU(6) sur les états physiques, on a appelé quarks constituants les états quarks appartenant à la représentation $\{6\}$ de $SU(6)_W^{class}$ et quarks courants les états de la représentation fondamentale de $SU(6)_W^{curr.}$.

$$\begin{cases} \text{quarks constituants} = \text{états quarks} \in \{6\} \text{ de } SU(6)_W^{class} \\ \text{antiquarks constituants} = \text{états antiquarks} \in \{\bar{6}\} \text{ de } SU(6)_W^{class} \end{cases}$$

$$\begin{cases} \text{quarks courants} = \text{états quarks} \in \{6\} \text{ de } SU(6)_W^{curr} \\ \text{antiquarks courants} = \text{états antiquarks} \in \{\bar{6}\} \text{ de } SU(6)_W^{curr} \end{cases}$$

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* On utilisera les définitions $\langle n, \lambda = \frac{1}{2} \mid Q_5^1 - i Q_5^2 \mid p, \lambda = \frac{1}{2} \rangle = G_A$

(λ = hélicité) $\langle n, \lambda = \frac{1}{2} \mid Q^1 - i Q^2 \mid p, \lambda = \frac{1}{2} \rangle = G_V$

Rappelons que les baryons $p, n \in \{56\}$ $L=0$ et aussi $p, n \in (8, \frac{1}{2})$

. La réduction de SU(6) par rapport à SU(3) x SU(2) entraine : $56 = (8, \frac{1}{2}) \oplus (10, \frac{3}{2})$.

L'hypothèse d'une transformation unitaire U entre ces 2 algèbres a été tôt admise : $Q^{class} = U\, Q^{curr}\, U^{-1}$, ce qui entraîne naturellement :

$$|q\ const^t\rangle = U\,|q\ curr^t\rangle$$

3. Transformation B.K.S.C.S. - M.

Partis de bases phénoménologiques, Buccella et al proposèrent pour cette transformation unitaire $U = \exp\left[i\theta\,(\vec{W}\times\vec{M})_z\right]$ les composantes du 3-vecteur $\vec{W}$ appartenant à une représentation irréductible {35} de SU(6) et $\vec{M}$ étant un vecteur sous O(3). C'est cette forme que retrouvera Melosh, lequel partit d'abord de l'hamiltonien du champ de quarks libres H . Obtenant un $SU(6)_W$ de charges après action de transformations infinitésimales sur le champ de quarks et application du théorème de Noether, il remarqua que, bien que $\left[H, SU(6)_W^{curr}\right] \neq 0$ une transformation de Foldy-Wouthuysen sur $H \rightarrow U\,H\,U^{-1}$ entrainait

$$\left[U\,H\,U^{-1}, SU(6)_W^{curr}\right] = 0 \quad \text{et par suite} \quad \left[H, U\,SU(6)_W^{curr}\,U^{-1}\right] = 0$$

$U\,SU(6)_W^{curr}\,U^{-1}$ est alors une algèbre de symétrie exacte de H et donc algèbre du groupe SU(6) de classification. En fait, l'équation précédente n'admet pas une seule solution en U , mais toute une classe de solutions. L'utilisation des charges lumière et une étude du spin des quarks conduira alors à caractériser l'unique "bonne" transformation parfois appelée la "seconde transformation de Melosh", que nous retrouverons de façon simple dans le Chapitre III. L'opérateur U a la forme pré-citée, la différence réside dans le fait que chez Melosh θ est un opérateur, tandis que chez Buccella et al, il n'est qu'un scalaire. Rappelons cependant que l'utilisation de U avec θ comme scalaire a conduit à de nombreux calculs phénoménologiques bien vérifiés expérimentalement.

4. Applications [9]

L'application la plus importante de l'opérateur de mixing U est le calcul des éléments de matrice de charge entre états physiques hadroniques: $\langle x | Q^{cur} | y \rangle$. En effet, l'action des générateurs de classification sur les états hadroniques pouvant être calculée, la connaissance de l'action de U sur ces mêmes états permet d'en déduire les quantitiés : $\langle x | Q^{cur} | y \rangle = \langle x | U^{-1} Q^{class} U | y \rangle$. Il s'ensuit la possibilité de déterminer des largeurs de désintégration, des contributions de résonances intermédiaires dans certaines réactions, etc...

Un autre type d'application intéressante est le calcul de formules de masse pour les hadrons à partir de la formule de S. Weinberg [10]

$$\left[Q_s^+, \left[Q_s^+, m^2\right]\right] = 0 \qquad Q_s^+ = Q_s^1 + i\, Q_s^2 .$$

Cette formule se transforme immédiatement suivant $\left[UQ_s^+U^{-1}, \left[UQ_s^+U^{-1}, Um^2U^{-1}\right]\right] = 0$ laquelle fournit la structure tensorielle de l'opérateur de masse transformé $\tilde{m}^2 = U m^2 U^{-1}$ par rapport au sous-groupe SU(3) x SU(3) du groupe de classification. L'utilisation du théorème Wigner-Eckart permet alors d'exhiber des relations entre éléments de matrice de l'opérateur $\tilde{m}^2$ entre états hadroniques $\langle x | \tilde{m}^2 | y \rangle$, d'où peuvent être déduites des formules de masse.

APPENDICE : CLASSIFICATION DES HADRONS

La réduction des représentations de SU(6) par rapport aux représentations de son sous-groupe SU(3) xSU(2) entraîne :

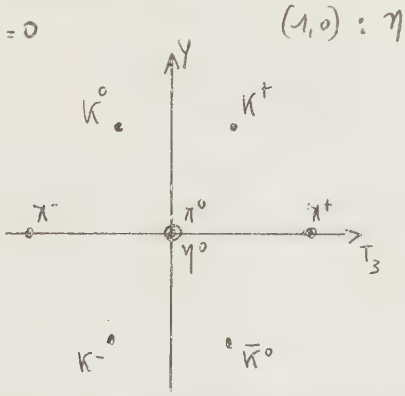
$$\{6\}\times\{6\}\times\{6\}\begin{cases}\{56\} = (8,\tfrac{1}{2}) + (10,\tfrac{3}{2})\\ \{70\} = (8,\tfrac{1}{2}) + (8,\tfrac{3}{2}) + (10,\tfrac{1}{2}) + (1,\tfrac{1}{2})\\ \{20\} = (8,\tfrac{1}{2}) + (1,\tfrac{3}{2})\end{cases}$$

$$\{6\}\times\{\bar{6}\}\begin{cases}\{35\} = (1,1) + (8,0) + (8,1)\\ \{1\} = (1,0)\end{cases}$$

Des considérations phénoménologiques conduisent à associer :

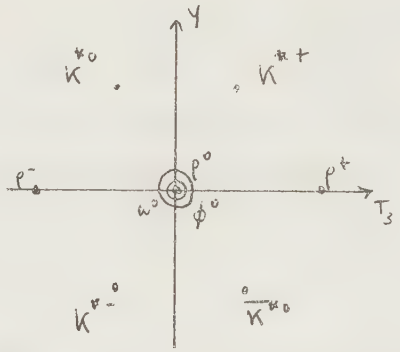
1. Mésons

$L=0$ $\qquad (1,0) : \eta'$



Octet des mésons pseudoscalaires (0^-)

$(8,0)$



Nonet de mésons vectoriels (1^-)

$(8 \oplus 1,1)$

$L = 1$: J peut être alors = 0,1,2 . Certains états sont manquants (notés ?), d'autres sont candidats à des résonances (notés [ ]) ;

$(8\oplus 1,0)$: $B^{\pm}_{0}$, $Q^{\pm}_{B}$, $\bar{Q}^{\bar{\pm}}_{B}$, ?, ? ; $(8\oplus 1,1)$ $J=2$: $A^{\pm}_{0}$, $K^{**\pm}_{0}$, $\bar{K}^{**\bar{\pm}}_{0}$, f, f'.

$J=1$: $[A^{\pm}_{1}]$, $Q^{\pm}_{A}$, $\bar{Q}^{\bar{\pm}}_{A}$, $D, [E]$.

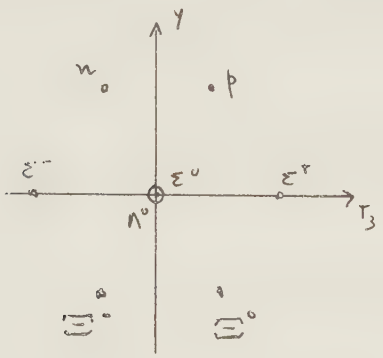
$J=0$ $\delta^{\pm}_{0}$, $[K^{\pm}_{0}, \bar{K}^{\bar{\pm}}_{0}]$, $[\varepsilon]$, $[S]$

Pour les masses et autres propriétés de ces résonances, voir Ref. [11]

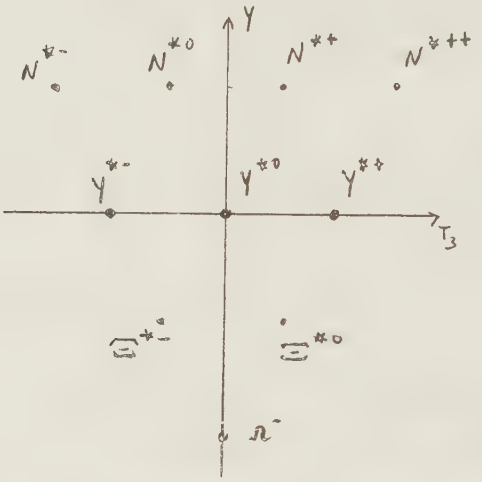
Pour $L \geqslant 2$, la situation est mons claire (voir Ref. [12]).

2. Baryons

56, L = 0 : on reconnaît



Octet de baryons stables
$(8, \frac{1}{2})$



Décuplet de baryons
$(10, \frac{3}{2})$

Des résonances sont classées dans une (70, L = 1) et dans une (56, L = 2). Certaines résonances récemment découvertes ont été associées à une (70, L = 2) et une (20, L = 1).

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CHAPITRE II

Current Quarks, Constituent Quarks and the Poincaré Group.

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A substantial progress has been achieved in the interpretation of the transformation connecting the axial charges to the corresponding generators of the collinear classification group ([1,2]).

It has been pointed out by MELOSH ([3]) that even in free-field theory an operator is needed to obtain from the SU_{6W} equal-time minimal charges a symmetry group of the Hamiltonian which is invariant under Lorentz boosts in the z-direction. However, the problem of constructing an SU_{6W} of operators commuting with both the Hamiltonian and the z-boosts has not a unique solution in the free-quark model ([4]). For example, the explicitly z-boost–invariant lightlike charges ([5]), which in the free-quark model are written as

$$\hat{Q}_\mu^\alpha = \int \mathrm{d}^4x\, \delta(x^0 + x^3)\, \psi^\dagger(x)(1 + \alpha_3)\, \frac{\lambda^\alpha \hat{\sigma}_\mu}{2}\, \psi(x)\,, \tag{1}$$

$$\hat{\sigma}_0 = I\,, \qquad \hat{\sigma}_3 = \sigma_3\,, \qquad \hat{\sigma}_{1,2} = \beta\sigma_{1,2} \qquad \alpha = 0, 1, \ldots, 8\,,$$

are symmetries of the free-quark Hamiltonian (in the equal-mass case), so that one might be disposed to identify them with the generators of the classification group. Again,

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MELOSH has shown (6) that this identification cannot be maintained when wave packets containing a (free) quark-antiquark pair (« mesons ») or three (free) quarks (« baryons ») are considered: in fact, the lightlike charges have complicated angular properties (1) and, therefore, some transformation is needed to obtain the classification charges, much simpler from this point of view.

The relevancy of the Poincaré group in understanding this transformation has been recently stressed by CELEGHINI and SORACE (7). They find an expression for the axial charges from the relativistic spin operator of a system of free particles (8). Therefore, they insist on the kinematical character of the transformation. The hypothesis that the classification mixing arises from the relativistic motion of the quarks in the hadrons has led LE YAOUANC *et al.* (9) to get successful predictions for the chiral content of the baryon octet.

The purpose of this paper is to derive the unitary transformation between the lightlike charges and the classification operators directly from the study of the representations of the Poincaré group, always in the case of free quarks. Starting from the properties of the lightlike charges, one finds that the mixing operator appears in the reduction of the product of two irreducible representations of the Poincaré group. Then it is shown, in a very transparent way, that the « current quarks », which behave as the fundamental representation of the lightlike-charge algebra, are related to the « constituent quarks », which correspond to the fundamental representation of the classification group, by a simple Wigner rotation (6,7).

To this extent one considers the 35 lightlike charges, $\hat{Q}^\alpha_\mu$ ($\mu = 0, 1, 2, 3$; $\alpha = 0, 1, \ldots, 8$), which are supposed to close an SU_{6W}-algebra, as suggested by their realization in the free-quark model eq. (1). The following *kinematical* property results from the definition of the lightlike charges: their SU_{6W}-algebra is unaffected by the transformations of the lightlike subgroup of the Poincaré group (which leaves the lightlike surface $x^0 + x^3 = 0$ invariant) generated by the following operators: $P_1 = P^0 + P^3$, P^1, P^2, $K_3 = M^{03}$, $E_1 = M^{01} + M^{31}$, $E_2 = M^{02} + M^{32}$ and $J_3 = M^{12}$. More precisely, we have the commutation relations

$$(2) \quad \begin{cases} [P_1, Q^\alpha_\mu] = [P^i, Q^\alpha_\mu] = [E_i, Q^\alpha_\mu] = [K_3, Q^\alpha_\mu] = [J_3, Q^\alpha_0] = [J_3, Q^\alpha_3] = 0 \\ \text{and} \\ [J_3, Q^\alpha_i] = i\varepsilon_{ij3} Q^\alpha_j \qquad (i = 1, 2)\,. \end{cases}$$

Also, from the first commutator in eq. (2) $[P_1, Q^\alpha_\mu] = 0$ one infers that the lightlike charges annihilate the vacuum state (5), $Q^\alpha_\mu|0\rangle = 0$, provided there are no massless particles. Nevertheless, this SU_{6W} is not invariant under the whole Poincaré group, unless the charges are conserved, *i.e.* the corresponding currents are divergenceless (which is the case in the free-quark model with equal masses). As a matter of fact, the commutators of the lightlike charges with the operators $P^0 - P^3$, $M^{01} - M^{31}$, $M^{02} - M^{32}$ are expected to be very complicated in general, being responsible for the intricacy of the so-called angular conditions.

From these considerations it is straightforward to conclude which is the most appropriate definition of states for dealing with lightlike charges. A vector of an irre-

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ducible representation of the Poincaré group $|pjj_3\rangle$ is such that

$$(3)\qquad \begin{cases} P^\mu|pjj_3\rangle = p^\mu|pjj_3\rangle & (p^2 = m^2)\,, \\ W^\mu W_\mu|pjj_3\rangle = -m^2 j(j+1)|pjj_3\rangle\,, \\ S_3(p)|pjj_3\rangle = j_3|pjj_3\rangle \end{cases}$$

with $S_3(p)$ being defined in terms of the Pauli-Lubanski operator

$$W_\mu = -\tfrac{1}{2}\varepsilon_{\mu\nu\varrho\sigma}P^\nu M^{\varrho\sigma}\,,$$

as $S_i(p) = L^\mu_i(p)W_\mu$; $L(p)$ is the particular Lorentz transformation chosen to reach the state $|pjj_3\rangle$ by starting from the rest state $|\mathring{p}jj_3\rangle$. The choice of the spin operator $S_i(p)$ is made in such a way that the matrix elements of the lightlike charges $\langle p'j'j'_3|\hat{Q}^\alpha_\mu|pjj_3\rangle$ are independent of $p'^0 + p'^3 = p^0 + p^3$, $p'^1 = p^1$ and $p'^2 = p^2$. In order to fulfil this property, the Lorentz transformation $L(p)$ is taken within the invariance Lorentz subgroup of the lightlike charges:

$$(4)\qquad L(p) = \exp\left[-\frac{i}{p_+}(p_1E_1 + p_2E_2)\right]\exp\left[-i\ln\frac{p_+}{m}K_3\right] \qquad (p_+ = p^0 + p^3)\,.$$

Let us consider now the free-quark model and the charges given in eq. (1). In this case, the spin operator $S_3(p)$, as we have defined it, is just the $\hat{Q}^0_3$ charge; and it is easily checked that the quark states $|p, j = \frac{1}{2}, j_3 = s_3 = \pm\frac{1}{2}, \beta\rangle_q$ (where we have added an SU_3 index $\beta = 1, 2, 3$) are *classified* by the SU_{6W} charges of the model, eq. (1), in the **6**-representation

$$(5a)\qquad \hat{Q}^\alpha_\mu|p\tfrac{1}{2}s_3\beta\rangle_q = \sum_{s'_3\beta'}\left(\frac{\lambda^\alpha\sigma_\mu}{2}\right)^{s_3\beta}_{s'_3\beta'}|p\tfrac{1}{2}s'_3\beta'\rangle_q\,.$$

Conversely, the antiquarks will belong to a $\bar{\mathbf{6}}$-representation

$$(5b)\qquad \hat{Q}^\alpha_\mu|p\tfrac{1}{2}s_3\beta\rangle_{\bar{q}} = \varepsilon_\mu\sum_{s'_3\beta'}\left(\frac{\overline{\lambda^\alpha\sigma_\mu}}{2}\right)^{s_3\beta}_{s'_3\beta'}|p\tfrac{1}{2}s'_3\beta'\rangle_{\bar{q}}\,,$$

where $\varepsilon_0 = \varepsilon_3 = -\varepsilon_1 = -\varepsilon_2 = 1$ and $(\overline{\lambda^\alpha\sigma_\mu})$ is the SU_6 6-dimensional-representation conjugate to $(\lambda^\alpha\sigma_\mu)$. Because of this classification property of the lightlike charges and their invariance under K_3-boosts, it has been pointed out by several people [4] that there are no *a priori* reasons for the distinction between current and constituent quarks in the free-quark model.

However, if one considers a (free) quark-antiquark state of the form $|p_1\tfrac{1}{2}s_{13}\beta_1\rangle_q \otimes$ $\otimes\,|p_2\tfrac{1}{2}s_{23}\beta_2\rangle_{\bar{q}}$, it will be classified by the lightlike charges in a $\mathbf{6}\otimes\mathbf{6} = \mathbf{35}\oplus\mathbf{1}$ representation of SU_{6W}. But this state corresponds in general to a highly reducible representation of the Poincaré group; instead, a « meson » state should be associate to the appropriate superposition which belongs to an irreducible representation of the Poincaré group, corresponding to an eigenstate of the operators P^μ, $P^2 = M^2$, W^2 and $L^\mu_3(P)W_\mu$:

$$(6)\qquad |Pjj_3;\, LS;\, \beta_1\beta_2\rangle = \int\frac{d^3p_1}{2p_1^0}\frac{d^3p_2}{2p_2^0}\sum_{s_{13},s_{23}}\langle p_1\tfrac{1}{2}s_{13};\, p_2\tfrac{1}{2}s_{23}|Pjj_3;\, LS\rangle|p_1\tfrac{1}{2}s_{13}\beta_1\rangle_q|p_2\tfrac{1}{2}s_{23}\beta_2\rangle_{\bar{q}}\,.$$

We stress the fact that this state, which we shall call « meson », is defined according to the prescription of eqs. (3), (4). The additional quantum numbers resolving the j-degeneracy, L and S, have been chosen in order to reproduce the classification scheme of $SU_6 \times O_3$, a point which will be better clarified later. The Clebesh-Gordan coefficients of the Poincaré group appearing in eq. (6) are given, *e.g.*, in ref. [10]:

$$(7) \qquad \langle p_1 \tfrac{1}{2} s_{13};\, p_2 \tfrac{1}{2} s_{23} | P j j_3;\, LS \rangle = \sqrt{8} \left(\frac{M^2}{M^2 - 4m^2} \right)^{\frac{1}{4}} \cdot$$

$$\cdot \delta^4(P - p_1 - p_2) \sum_{s'_{13}, s'_{23}} \mathscr{D}^{\frac{1}{2}}_{s_{13} s'_{13}} [L^{-1}(p_1) L(p_1 \leftarrow P) L(P)]\, \mathscr{D}^{\frac{1}{2}}_{s_{23} s'_{23}} [L^{-1}(p_2) L(p_2 \leftarrow P) L(P)] \cdot$$

$$\cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & | & S \\ s'_{13} & s'_{23} & | & S_3 \end{pmatrix} \begin{pmatrix} S & L & | & j \\ S_3 & L_3 & | & j_3 \end{pmatrix} y^L_{L_3}(\hat{q}), \qquad q_i = L^\mu_i(P) \frac{(p_1 - p_2)_\mu}{2}, \quad \hat{q} = q / \sqrt{-q^2},$$

where the Lorentz transformations $L(p_1)$, $L(p_2)$ and $L(P)$ are defined in eq. (4) and $L(p_i \leftarrow P)$ $(i = 1, 2)$ is the pure Lorentz transformation in the 2-plane (p_i, P) taking the unit vector P^μ/M into the unit vector p_i^μ/m.

From eqs. (6), (7) a classification group $U_6 \times U_6 \times O_3$ is easily introduced for these « meson » states, which can be written as

$$(8) \qquad |P j j_3;\, LS;\, \beta_1 \beta_2 \rangle = \text{const} \cdot \int \mathrm{d}\hat{q}\, y^L_{L_3}(\hat{q}) \cdot$$

$$\cdot \sum_{s_{13}, s_{23}} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & | & S \\ s_{13} & s_{23} & | & S_3 \end{pmatrix} \begin{pmatrix} S & L & | & j \\ S_3 & L_3 & | & j_3 \end{pmatrix} |p_1(P) s_{13} \beta_1\rangle_q |p_2(P) s_{23} \beta_2\rangle_q,$$

whenever the new quark and antiquark states are defined in terms of the previous ones as follows:

$$(9) \qquad |p(P) s_3 \beta\rangle_{q,q} = R(p, P)_{s_3 s'_3} |p s'_3 \beta\rangle_{q,q}$$

with

$$R(p, P) = \mathscr{D}^{\frac{1}{2}}[L^{-1}(p) L(p \leftarrow P) L(P)].$$

The states in the l.h.s. of eq. (9) will be called « constituent » quarks (antiquarks), although they differ from the « current » quarks (antiquarks), in the r.h.s., only by a spin rotation $R(p, P)$. At this point, one can consider the « *classification charges* » $Q^\alpha_\mu(\bar{Q}^\alpha_\mu)$, under which constituent quarks (respectively, antiquarks) transform as a **6** (respectively, $\bar{\mathbf{6}}$) representation:

$$(10) \qquad \begin{cases} Q^\alpha_\mu |p(P) s_3 \beta\rangle_q = \sum_{s'_3 \beta'} \left(\dfrac{\lambda^\alpha \sigma_\mu}{2} \right)_{s_3 \beta, s'_3 \beta'} |p(P) s'_3 \beta'\rangle_q, \\ \bar{Q}^\alpha_\mu |p(P) s_3 \beta\rangle_{\bar{q}} = \sum_{s'_3 \beta'} \left(\dfrac{\lambda^\alpha \sigma_\mu}{2} \right)_{s_3 \beta, s'_3 \beta'} |p(P) s'_3 \beta'\rangle_{\bar{q}}. \end{cases}$$

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[10] P. MOUSSA and R. STORA: *Methods in subnuclear physics*, in *1966 Herceg-Novi International School*, Vol. **2** (New York, N. Y., 1968), p. 265.

In particular $Q_i^0(\bar{Q}_i^0)$ is the spin operator for constituent quarks (respectively, antiquarks). The operators $Q_\mu^\alpha \otimes I$, $I \otimes \bar{Q}_\mu^\alpha$ generate the classification group $U_{6q} \times U_{6\bar{q}}$, and eq. (8) implies that our « meson » states behave as members of a $(\mathbf{6}, \bar{\mathbf{6}})$-representation. Furthermore, we notice that the spin operator for the meson can be written after eq. (8) as (an obvious simplification has been introduced in the notation)

$$J_i = L_i^\mu(P) W_\mu = L_i + S_i = L_i + Q_i^0 + \bar{Q}_i^0 , \tag{11}$$

where $\boldsymbol{L}$ is the « internal orbital angular momentum » which is associated to the relative 3-momentum, $q_i = \frac{1}{2} L_i^\mu(P)(p_1 - p_2)_\mu$, and generates a group O_3 commuting with Q_μ^α and $\bar{Q}_\mu^\alpha$. The operator $\boldsymbol{S}$ is the « total quark spin » of the « meson », $S_i = Q_i^0 + \bar{Q}_i^0$. By summing up, the meson state is *classified* in a $(\mathbf{6}, \bar{\mathbf{6}})_L$-representation of the group $U_6 \times U_6 \times O_3$, generated by Q_μ^α and $\bar{Q}_\mu^\alpha$, whose fundamental representations are associated to the constituent quarks and antiquarks, defined in eq. (9), and to the orbital angular momentum L.

By comparing eqs. (5), (9), (10) we conclude that the classification and lightlike charges are related to each other as follows:

$$\hat{Q}_\mu^\alpha = R^{-1}(p, P)[Q_\mu^\alpha + \varepsilon_\mu \bar{Q}_\mu^\alpha] R(p, P) , \tag{12}$$

where $R(p, P)$ is the rotation of the quark (antiquark) spin defined in eq. (9), which applies to both the quark and antiquark states, and p is the quark (antiquark) momentum operator. Notice that everything is consistent with the properties of the different charges under the charge conjugation transformation which brings into relation the quark and antiquark states.

As a matter of fact, this $U_6 \times U_6 \times O_3$ is a collinear classification group, as it is invariant under K_3-boosts. Indeed, let us take meson collinear states such that $P^i = p_1^i + p_2^i = 0$, $i = 1, 2$. The particular choice we have made for $L(p)$ in eq. (4) implies

$$\begin{cases} R(p, P) = \mathscr{P}[L^{-1}(p) L(p \leftarrow P) L(P)] = \exp[-i\theta(p, P)] \dfrac{(\boldsymbol{\sigma} \times \boldsymbol{P})_3}{p_\perp} , \\ \theta(p, P) = \operatorname{arctg} \dfrac{p_\perp}{[(p^0 + p^3)/(P^0 + P^3)](M + m)} , \qquad p_\perp = \sqrt{(p_1)^2 + (p_2)^2} . \end{cases} \tag{13}$$

This Wigner rotation is just the transformation proposed by MELOSH ([6,7]). It commutes with the K_3-operator and that ensures the invariance of the classification under boosts in the z-direction. This may be verified as follows. Each « constituent » quark is a superposition of different current-quark spin states, as given in eq. (9). When we K_3-boost the state, this superposition is unchanged, as both $R(p, P)$ and the current-quark spin operator, $\hat{Q}_i^0$, commute with K_3, so that the constituent spin operator will be invariant under K_3-boosts. A similar analysis can be made for antiquark spin and these considerations are trivially extended to the SU_3 indices.

As a consequence, the use of $U_6 \times U_6 \times O_3$ as the *classification group for collinear states* is perfectly justified and it has led to some successfull predictions ([2]). However, if the meson states are required to correspond to irreducible representations of the Poincaré group, they are not simply classified in irreducible representations of the SU_6 generated by the lightlike charges, because of the Melosh transformation eqs. (9), (12), (13).

It is worth noticing that the mixing operator required in this simple model to relate the lightlike charges to the generators of the classification group has the same form already found in a phenomenological algebraic approach ([1,11]) to the saturation of current algebra: not only the free-quark model exhibits such a unitary transformation, but it seems to suggest one with the right properties. More success for the puzzling « naïve-quark model », where hadrons are made of a few freelike, independent quarks, and which works so well in deep inelastic processes, duality diagrams and many other phenomena. What cannot be neglected, however, is the relativistic character of the quark motion inside hadrons ([7,9])!

Recently an interesting formal treatment has been suggested ([12]), where quarks are kept inside hadrons by some sort of superselection rule (colour is not to be observed). Perhaps, the enigma of quarks, which behave as free components of hadrons although they are so bound that presumably they can hardly be observed, will find a rather simple solution: a genuine Columbus' egg!

* * *

We are grateful to H. RUEGG for very interesting discussions and for calling our attention to ref. ([10]).

([11]) F. BUCCELLA, E. CELEGHINI and C. A. SAVOY: *Nuovo Cimento*, **7** A, 281 (1972).
([12]) D. AMATI and M. TESTA: *Phys. Lett.*, **48** B, 227 (1974).

CHAPITRE III : CONCLUSION

L'étude précédente de la relation entre algèbre des charges et algèbre de classification lève en particulier tout mystère sur cette notion de "quark courant", notion utile simplement sur un plan calculatoire, mais opposée de façon un peu trop sophistiquée à celle de "quark constituant", la seule ayant une signification physique.

Néanmoins, depuis la parution de cet article, trois travaux [1], [2], [3] (au moins) ont paru, développant des arguments physiques analogues aux nôtres. Il paraît bon de discuter brièvement de leur approche légèrement différente, mais que les auteurs des réf. [1] et [2] disent en parfait accord avec la nôtre.

Une autre remarque qui pourrait avoir quelque intérêt est la correspondance entre état quark courant et état quark constituant à partir de la notion d'état de Wigner et de Mackey.

Enfin, nous terminerons cette discussion sur des considérations concernant les états à 3 particules (cas des baryons).

A) VARIABLES DU CENTRE DE MASSE POUR UN SYSTEME DE 2 PARTICULES RELATIVISTES

La transformation (singulière) de Gartenhaus et Schwartz (*) permet d'éliminer le mouvement du centre de masse (c.m.) et d'étudier le mouvement relatif dans le cas d'un système de particules non relativistes sans interactions. Pour 2 particules d'impulsion et position respectives $\vec{p}_1, \vec{p}_2, \vec{r}_1, \vec{r}_2$ telles que $[r_{\alpha i}, p_{\beta j}] = i \delta_{\alpha\beta} \delta_{ij}$ $(\alpha = 1,2 ; i,j = 1,2,3)$

(*) Cf. Réf. [4]

on peut considérer les variables "externes" $\vec{P} = \vec{p}_1 + \vec{p}_2$, $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$ et les variables "internes" : $\vec{k} = \frac{\vec{p}_1 - \vec{p}_2}{2}$, $\vec{r} = \vec{r}_1 - \vec{r}_2$. Définissant l'opérateur unitaire $U(\alpha) = \exp\left[i \frac{\alpha}{2} \left(\vec{P}.\vec{R} + \vec{R}\vec{P} \right) \right]$, on remarque que :

$$\lim_{\alpha \to \infty} U(\alpha)\, P\, U(\alpha)^* = 0 \qquad \text{et} \qquad \lim_{\alpha \to \infty} U^*(\alpha)\, R\, U(\alpha) = 0$$

tandis que $\lim_{\alpha \to \infty} U(\alpha)\, X\, U(\alpha)^* = X$ pour $X = \vec{k}, \vec{r}$ ou $\vec{s}_\alpha$

($\vec{s}_\alpha$ spin de la particule α).

Cette étude a été généralisée au cas d'un système de particules relativistes par H. Osborn [4] lequel a montré que le passage dans le référentiel du c.m. $\vec{P} = \vec{p}_1 + \vec{p}_2 = 0$ induisait une rotation du spin de chaque particule égale à $L^{-1}(p_\alpha)\, L(p_\alpha \leftarrow P)\, L(P)$ si on utilise les notations du chapitre précédent.

C'est cette approche qu'ont préférée les auteurs de la ref. [1] et de la réf. [3] (dont l'auteur est H. Osborn précité), et qui est aussi implicitement contenue dans la réf. [2] . L'avantage de cette étude est peut-être de mieux mettre en évidence le mouvement interne du système composé et par suite de mieux montrer les rotations du spin de chaque particule nécessaires pour pouvoir définir le moment cinétique total du système composé.

Cependant, ces remarques peuvent aussi être tirées de l'étude Moussa et Stora [5],[6] sur la réduction du produit de 2 représentations de Poincaré dans le cas du couplage L-S. En effet, dans la réf. [5], il est rappelé que en cinématique non relativiste une rotation R intéresse de la même manière les 3 moments cinétiques (spin $\vec{s}_1$ et $\vec{s}_2$ des particules et moment orbital $\vec{l}$) dont la somme fournit le moment cinétique total, car ils sont repérés par rapport aux mêmes axes. Mais dans le cas relativiste, les axes de référence diffèrent pour chacune des particules et le système composé, et par suite une rotation dans le système du c.m. se traduira sur les spins par 2 matrices $\mathcal{D}^{s_1}(R_1)$, $\mathcal{D}^{s_2}(R_2)$.

On saura se ramener à une situation analogue au couplage L-S non relativiste en choisissant des tétrades telles que $R_1 = R_2 = R$. Les coefficients de Clebsch-Gordan de Poincaré étant déterminés avec ce choix de tétrades particulières, le passage à des tétrades quelconques fait apparaître les rotations de spin $L^{-1}(p_\alpha)\, L(p_\alpha \leftarrow p)\, L(p)$.

Ainsi donc, dans chacune des deux approches, la difficulté du passage du cas non relativiste au cas relativiste apparaît clairement et se traduit en particulier par une rotation de spin de chacune des particules composantes.

Si l'approche de [1] est peut-être plus adaptée à une discussion éventuelle du problème d'interactions entre particules, elle a le défaut de mettre en jeu une transformation singulière et dont l'interprétation physique reste obscure. De plus, elle utilise quand même les résultats de [5],[6] pour la forme explicite des coefficients de Clebsch-Gordan de Poincaré.

B) ETATS DE MACKEY ET ETATS DE WIGNER

Nous nous intéressons à présent aux états d'une représentation irréductible du groupe de Poincaré.

Considérons les états de la particule au repos : les états $|\mathring{p}, \sigma_1, \sigma_3\rangle$ (en gardant les notations du chapitre précédent) peuvent être considérés comme formant une base de représentation du groupe des rotations

$$U(h)\,|\mathring{p}, \sigma_3\rangle = |\mathring{p}, \sigma_3'\rangle\, \mathcal{D}^{\Lambda}_{\sigma_3'\sigma_3}(h) \qquad \text{avec } h\mathring{p} = \mathring{p}$$

Posons alors

$$|\Lambda, \sigma_1, \sigma_3\rangle = U(\Lambda)\,|\mathring{p}, \sigma_1, \sigma_3\rangle \qquad \Lambda \in \mathcal{L} : \text{groupe de Lorentz}$$

et appelons état de Mackey un état ainsi défini.
On vérifie immédiatement :

$$U(\Lambda')\,|\Lambda, \sigma_1, \sigma_3\rangle = |\Lambda'\Lambda, \sigma_1, \sigma_3\rangle \qquad \Lambda, \Lambda' \in \mathcal{L}$$

$$** \qquad |\Lambda h, \sigma_1, \sigma_3\rangle = |\Lambda, \sigma_1, \sigma_3'\rangle\, \mathcal{D}^{\Lambda}_{\sigma_3'\sigma_3}(h) \qquad h\mathring{p} = \mathring{p}$$

Considérons à présent l'ensemble des transformations Λ_p de $\mathcal{L}$ telles que : $\Lambda_p \mathring{p} = p$, p quadrivecteur vérifiant $p^2 = \mathring{p}^2$. Remarquons que deux transformations Λ_p et Λ'_p ne diffèrent que par un élément h du petit groupe de p ; il est par suite naturel de considérer la relation d'équivalence

$$\Lambda \sim \Lambda' \iff \Lambda^{-1}\Lambda' \in SO(3)_{\mathring{p}} .$$

Fixons-nous alors un boost standard $L(p) : \mathring{p} \longrightarrow p$ (en choisissant par exemple la forme donnée dans l'équation (4) du

chapitre précédent, ou bien la forme utilisée par Jacob et Wick [7]. Utilisant les notations de la réf. [5],[6], nous écrirons :

$$L(p) = [p]$$

Posant

$$|[p], \lambda_1 \lambda_3\rangle = U([p]) \,|\, \mathring{p}, \lambda_1 \lambda_3\rangle$$

nous avons ainsi redéfini un état de Wigner.
Ainsi un état de Wigner apparaît comme un état de Mackey particulier, et plus précisément comme un représentant de classe d'équivalence correspondant à la relation d'équivalence définie ci-dessus. On remarque d'après la relation (* *) que les états de Mackey forment une base surcomplète d'états, ce qui n'est plus le cas des états de Wigner qui forment une base complète d'états.

On retrouve immédiatement [8] la relation générale liant états de Mackey et états de Wigner :

$$|\Lambda, \lambda_1 \lambda_3\rangle = |[p], \lambda_1 \lambda'_3\rangle \, \mathcal{D}^{s}_{\lambda'_3 \lambda_3}\left(L^{-1}(p)\Lambda\right) \qquad p = \Lambda \mathring{p}$$

L'utilisation du formalisme "à la Mackey" ou "à la Wigner" dépend du type de problème considéré [8], chaque sorte d'états ayant ses propres avantages.

Néanmoins, se ramenant à notre relation entre état quark courant et constituant

$$|L(p_\alpha \leftarrow P)\, L(P), \lambda_1 \lambda_3\rangle = |L(p_\alpha), \lambda_1 \lambda'_3\rangle \, \mathcal{D}^{s}_{\lambda'_3 \lambda_3}\left(L^{-1}(p_\alpha) L(p_\alpha \leftarrow P) L(P)\right) \qquad \alpha = 1,2.$$

nous pouvons dire que nous passons d'un état quark courant à un état quark constituant par un changement de "boost standard":
$L(p_\alpha) \longrightarrow L(p_\alpha \leftarrow P) L(P)$, ou ce qui est équivalent, d'un état de Mackey à un autre état de Mackey, ces 2 états appartenant à la même classe d'aquivalence. En tant qu'états de Wigner, nous avons fait un changement de base.

C) LES ETATS A 3 QUARKS (BARYONS)

La réduction du produit de 3 représentations de Poincaré a été étudiée par de nombreux auteurs (cf. en particulier réfs [5] et [6]). Nous proposons ici une réduction avec couplage L-S, analogue à celle utilisée dans le chapitre précédent pour 2 particules, et généralisant en fait aux baryons les résultats obtenus pour les mésons. Néanmoins, nous verrons que l'étude qui suit présente certaines difficultés et suggère d'autres développements.

Les notations utilisées sont celles des réfs [5] et [6](*). Un état à 3 particules sera noté :

$$\left|[p_1][p_2][p_3]\,\sigma_1\sigma_2\sigma_3\right\rangle = \bigotimes_{i=1}^{3} \left|[p_i]\,\sigma_i\right\rangle$$

et une transformation de Poincaré est représentée par

$$U(a,\Lambda)\left|[p_1][p_2][p_3]\,\sigma_1\sigma_2\sigma_3\right\rangle = \prod_{i=1}^{3} \mathcal{D}^{\Delta_i}_{\sigma_i'\sigma_i}\left([\Lambda p_i]^{-1}\Lambda[p_i]\right) e^{i\Lambda p_i\cdot a}\left|[\Lambda p_1][\Lambda p_2][\Lambda p_3]\,\sigma_1'\sigma_2'\sigma_3'\right\rangle$$

Les états d'une représentation irréductible $\left|[P]\,j\,\mu\right\rangle$ peuvent être définis[5] par l'application de l'opérateur de "projection"(**)

$$\mathbb{P}^{M,j}_{\rho,\mu} = \int \frac{da\,d\Lambda}{(2\pi)^4}\; \mathcal{D}^{M,j\,*}_{\rho,\mu,\rho,\mu}(a,\Lambda)\, U(a,\Lambda)$$

avec :

$$\mathcal{D}^{M,j\,*}_{\rho,\mu,\rho,\mu}(a,\Lambda) = 2P^0\, e^{i P\cdot a}\,\delta\left(\vec{P} - \vec{\Lambda}\vec{P}\right)\mathcal{D}^{j}_{\mu\mu}\left([P]^{-1}\Lambda[P]\right)$$

(*) Nous utiliserons la notation $[p]$ pour l'opérateur $L(p)$ du chapitre précédent $[p]\,\mathring{p} = p$. A l'intérieur d'un ket $\left|[P], s, \sigma\right\rangle$, $[P]$ représentera la tétrade associée à p.

(**) On peut vérifier par le calcul que, imposer à l'état $|\phi\rangle$ de se comporter comme l'état $\left|[P], j, \mu\right\rangle$ de la représentation irréductible (M, j) de Poincaré, est équivalent à lui faire subir l'action de l'opérateur $\mathbb{P}^{M,j}_{\rho,\mu}$.

sur l'état le plus général :

$$|\phi\rangle = \sum_{\{\sigma_i\}} \int \prod_{i=1}^{3} \frac{d^3 p_i}{2\omega_i} \varphi_{\{\sigma_i\}}\left([p_1],[p_2],[p_3]\right) |[p_1][p_2][p_3]\,\sigma_1\sigma_2\sigma_3\rangle$$

avec : $\omega_i = \sqrt{\vec{p}_i^{\,2} + m_i^2}$

On obtient ainsi :

$$\mathbb{P}^{M,\gamma}_{P,\mu} |\phi\rangle = \int \prod_{i=1}^{3} \frac{d^3\vec{p}_i}{2\omega_i} \delta\left(P - \sum_{i=1}^{3} p_i\right) \times \int dR\, \mathcal{D}^{j*}_{\mu\mu}(R) \prod_{i=1}^{3} \mathcal{D}^{j_i}_{\sigma_i'\sigma_i}(R_i) \times \varphi\left([\Lambda^{-1}p_1][\Lambda^{-1}p_2],[\Lambda^{-1}p_3]\right)$$
$$|[p_1][p_2][p_3]\,\sigma_1'\sigma_2'\sigma_3'\rangle$$

avec : $R_i = [p_i]^{-1} \Lambda [\Lambda^{-1} p_i]$

et $\Lambda = [P]\, R\, [P]^{-1}$ où R appartient au petit groupe de $\overset{o}{P}$.

Il est possible d'avoir $R_1 = R_2 = R_3 = R$ (voir §A de ce même chapitre) en choisissant les tétrades $[p_i]$ telles que :

$$[p_i] = L(p_i \leftarrow P)\,[P] \qquad i = 1,2,3.$$

$L(p_i \leftarrow P)$ étant la transformation de Lorentz plane transformant P en un vecteur parallèle à p_i [x]. Ainsi :

$$\mathbb{P}^{M,\gamma}_{P,\mu} |\phi\rangle = \int \prod_{i=1}^{3} \frac{d^3\vec{p}_i}{2\omega_i} \delta\left(P - \sum_{i=1}^{3} p_i\right) \times \int dR\, \mathcal{D}^{j*}_{\mu\mu}(R) \prod_{i=1}^{3} \mathcal{D}^{j_i}_{\sigma_i'\sigma_i}(R)$$

$$\varphi_{\sigma_1\sigma_2\sigma_3}\left([\Lambda^{-1}p_1][\Lambda^{-1}p_2],[\Lambda^{-1}p_3]\right) |[p_1][p_2][p_3]\,\sigma_1'\sigma_2'\sigma_3'\rangle$$

Utilisant la méthode de couplage de 3 moments angulaires proposée par Chakrabarti [3], nous obtenons :

$$\prod_{i=1}^{3} \mathcal{D}^{j_i}_{\sigma_i'\sigma_i}(R) = \sum_{\Lambda,\Sigma} A^{*}\begin{pmatrix} \Lambda & \sigma & \Sigma \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} A\begin{pmatrix} \Lambda & \sigma' & \Sigma \\ \sigma_1' & \sigma_2' & \sigma_3' \end{pmatrix} \mathcal{D}^{\Lambda}_{\sigma'\sigma}(R)$$

$\sigma = \sigma_1 + \sigma_2 + \sigma_3$

$\sigma' = \sigma_1' + \sigma_2' + \sigma_3'$

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[x] On vérifiera aisément ce résultat grâce à la propriété de covariance de $L(p \leftarrow P)$: $L(\mathcal{R}p_i \leftarrow \mathcal{R}P) = \mathcal{R}\, L(p_i \leftarrow P)\, \mathcal{R}^{-1} \quad \forall \mathcal{R} \in \text{Lorentz}$.

les coefficients A étant définis par

$$| \Lambda_1 \sigma_1 \zeta \; ; \Lambda_2 \sigma_2 \Lambda_3 \rangle = \sum_{\sigma = \Lambda_1 + \Lambda_2 + \Lambda_3} A \begin{pmatrix} \Lambda & \sigma & \zeta \\ \Lambda_1 & \Lambda_2 & \Lambda_3 \end{pmatrix} | \Lambda_1 \sigma_1 ; \Lambda_2 \sigma_2 ; \Lambda_3 \sigma_3 \rangle$$

L'avantage de cette méthode est d'obtenir des nombres quantiques $\Lambda_1 \sigma_1 \zeta$ (ζ jouant le rôle de paramètre de dégénérescence) ne dépendant pas de l'ordre dans lequel est effectué le produit des 3 représentations. Notons que les coefficients A sont complexes en général et tels que $A^*(\zeta) = A(-\zeta)$.

Portant ce résultat dans l'expression de $P^{M,\delta}_{p,\mu} | \phi \rangle$ et posant :

$$\begin{cases} \varphi_{\sigma_1 \sigma_2 \sigma_3}\left([\Lambda^{-1} p_1], [\Lambda^{-1} p_2], [\Lambda^{-1} p_3]\right) A^* \begin{pmatrix} \Lambda & \sigma & \zeta \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = \psi_\sigma^{\Lambda_1 \zeta}\left([\Lambda^{-1} p_1], [\Lambda^{-1} p_2], [\Lambda^{-1} p_3]\right) \\ | [p_1][p_2][p_3] \sigma_1 \sigma_2 \sigma_3 \rangle A \begin{pmatrix} \Lambda & \sigma & \zeta \\ \sigma_1 & \sigma_2 & \sigma_3 \end{pmatrix} = | [p_1][p_2][p_3] \Lambda_1 \zeta_1 \sigma \rangle \end{cases}$$

on obtient :

$$P^{M,\delta}_{p,\mu} | \phi \rangle = \int \prod_{i=1}^{3} \frac{d^3 \vec{p}_i}{2\omega_i} \delta\left(p - \sum_{i=1}^{3} p_i\right) \times \int dR \; \psi_\sigma^{\Lambda_1 \zeta}\left([\Lambda^{-1} p_1], [\Lambda^{-1} p_2], [\Lambda^{-1} p_3]\right)$$
$$\times \mathcal{D}^{\Lambda}_{\sigma\sigma'}(R) \, \mathcal{D}^{\zeta *}_{\mu\mu'}(R) \, | [p_1][p_2][p_3] \Lambda_1 \zeta \sigma' \rangle$$

avec $\Lambda = [p] R [p]^{-1}$

Interessons-nous à présent à la fonction $\psi_\sigma^{\Lambda_1 \zeta}\left([\Lambda^{-1} p_1], [\Lambda^{-1} p_2], [\Lambda^{-1} p_3]\right)$

Etant donnés 3 quadrivecteurs p_i^s vérifiant $\sum_{i=1}^{3} p_i^s = \breve{p}$,

on peut toujours trouver une rotation R' appartenant au petit groupe de $\overset{o}{P}$ telle que :

$$p_i = [p] R' p_i^s$$

$\psi_\sigma^{s,\tau}$ Grâce à ce "changement de variables", on peut considérer comme fonction de $R^{-1}R'$ et $\mathcal{J}$, $\mathcal{J}$ représentant les variables du "diagramme de Dalitz". Remarquons que ces dernières peuvent s'exprimer en fonction des invariants $s_{ij} = (p_i + p_j)^2$: on n'a en fait que 2 variables indépendantes, la masse totale étant fixée.

On posera donc :

$$\psi_\sigma^{s,\tau}\left([\Lambda^{-1}p_1],[\Lambda^{-1}p_2],[\Lambda^{-1}p_3]\right) = \eta_\sigma^{s,\tau}(R^{-1}R', \mathcal{J})$$

Développant η suivant les rotations, il vient :

$$\eta_\sigma^{s,\tau}(R^{-1}R', \mathcal{J}) = \chi_\sigma^{s,\tau;\,\ell,\nu,\nu'}(\mathcal{J})\, \mathcal{D}^{\ell}_{\nu\nu'}(R'^{-1}R)$$

$$= \chi_\sigma^{s,\tau;\,\ell,\nu,\nu'}(\mathcal{J})\, \mathcal{D}^{\ell}_{\nu\alpha}(R'^{-1})\, \mathcal{D}^{\ell}_{\alpha\nu'}(R)$$

Et ainsi, en utilisant les propriétés des matrices $\mathcal{D}^j$:

$$\mathbb{P}^{M,\gamma}_{p,\mu}|\Phi\rangle = \int \prod_{i=1}^{3} \frac{d^3\vec{p}_i}{2\omega_i}\, \delta\left(P - \sum_{i=1}^{3} p_i\right) \times \chi_\sigma^{s,\tau;\,\ell,\nu,\nu'}(\mathcal{J})\, \mathcal{D}^{\ell}_{\nu\alpha}(R'^{-1})$$

$$\times \begin{pmatrix} \ell & s \\ \alpha & \sigma' \end{pmatrix} \begin{pmatrix} \ell & s \\ \nu & \sigma \end{pmatrix} \begin{pmatrix} j \\ \mu \end{pmatrix} |[p_1][p_2][p_3]\; s,\tau,\sigma'\rangle$$

On obtient ainsi :

$$\mathbb{P}^{M,\gamma}_{p,\mu}|\Phi\rangle = \sum_{\ell,\nu;\,s,\tau} \int d\mathcal{J}\, \Gamma^{j,\mu;\,\ell,\nu;\,s,\tau}(\mathcal{J})\, |[p]\; j,\mu;\,\ell,\nu;\,s,\tau;\,\mathcal{J}\rangle$$

en posant :

$$\Gamma^{j,\mu;\,\ell,\nu;\,s,\tau}(\mathcal{J}) = \chi_\sigma^{s,\tau;\,\ell,\nu,\nu'}(\mathcal{J}) \cdot \begin{pmatrix} \ell & s \\ \nu' & \sigma \end{pmatrix} \begin{pmatrix} j \\ \mu \end{pmatrix}$$

et

$$\left| [P], j, \mu ; \ell, \nu ; s, \zeta ; \mathcal{J} \right\rangle = N \int \prod_{i=1}^{3} \frac{d^3 \vec{p}_i}{2\omega_i} \, \delta\left(P - \sum_{i=1}^{3} p_i\right) \times$$

$$\times \, \delta\left(\mathcal{J}(p_1, p_2, p_3) - \mathcal{J}\right) \times \mathcal{D}^{\ell}_{\nu\alpha}(R'^{-1}) \times \left(\begin{matrix} \ell & s \\ \alpha & \sigma' \end{matrix} \middle| \begin{matrix} j \\ \mu \end{matrix} \right) \left| [p_1][p_2][p_3] \, s \, \zeta \, \sigma' \right\rangle$$

(N étant un facteur de normalisation à calculer à partir du produit scalaire de 2 états).

En résumé, les nombres quantiques spécifiant l'état sont

α) les quantités m_1, s_1 ; m_2, s_2 ; m_3, s_3 spécifiant les représentations des 3 particules constituantes.

β) les 4 composantes de P , la quantité j caractérisant le moment cinétique total, et la projection de ce moment cinétique sur l'axe[x] $n_3(P) = [P] \, n_3(\mathring{P})$.

γ) les nombres quantiques "de dégénérescence" :
s représentant le spin total, la dégénérescence de s étant elle même levée par le nombre quantique ζ ;
ℓ caractérisant le moment orbital et ν la projection du moment orbital sur l'axe $\Lambda' n_3(\mathring{P}) = [P] R' n_3(\mathring{P})$;
enfin les "variables" de Dalitz $s_{ij} = (p_i + p_j)^2$ au nombre de trois, mais reliées à la masse totale $P^2 = M^2$, et ainsi fournissant deux variables indépendantes.

Il est intéressant de remarquer que les nombres quantiques $P ; j, \mu ; \ell, \nu ; s, \zeta$ peuvent être considérés comme "démocratiques" [11] , les opérateurs dont ils sont valeurs propres étant invariants (au signe prés[xx]) sous l'action des permutations des 3 particules. Seules les variables de Dalitz ne satisfont pas à cette propriété. Cependant on remarque qu'une permutation ne peut que transformer une variable

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[x] La tétrade associée au vecteur p , du genre temps étant caractérisée par 3 vecteurs $n_i(p)$ $i = 1,2,3$ satisfaisant à : $n_i(p) \cdot n_j(p) = -\delta_{ij}$
$p \cdot n_i(p) = 0$ et $\det\left(p, n_1(p), n_2(p), n_3(p)\right) = 1$.

[xx] En effet, le nombre quantique ζ provient de l'opérateur $Z = (\vec{s}_1 \times \vec{s}_2) \cdot \vec{s}_3$ lequel est invariant sous une permutation circulaire, mais change de signe sous une transposition.

Λ_{ij} en un autre $\Lambda_{i'j'}$. On peut dire par suite que l'ensemble est "globalement démocratique".

D'autre part, les états à 3 particules ci-dessus ont été calculés à partir de tétrades $[p_i]$ particulières $[p_i] = L(p_i \leftarrow P)[P]$. Le passage à des tétrades quelconques remplace chaque état $|[p_i], \sigma_i\rangle$ en $\mathcal{D}^{s_i}_{\sigma_i \tau_i}\left([p_i]^{-1} L(p_i \leftarrow P)[P]\right) |[p_i]\ \tau_i\rangle \quad i = 1,2,3.$
Nous retrouvons ainsi pour les baryons, les rotations de spin faisant passer d'un "état quark courant" à un "état quark constituant".

Il est à noter que ces rotations de Wigner apparaissant naturellement dans l'étude du couple $L - S$ d'un système de particules.

Enfin, on peut remarquer que la fonction $\psi^{\Delta,S}_{\sigma}$ introduite dans le cours du calcul ci-dessus est définie sur une surface fermée à 5 dimensions, surface qui peut être amenée par déformation radiale sur la sphère de rayon unité S^5 [10]. Se rappelant l'étude de Dragt [11] concernant la classification des états à 3 particules sans spin non relativistes suivant les représentations de SU(3), il apparaît prometteur de développer la fonction $\psi^{\Delta,S}_{\sigma}$ suivant les fonctions propres de SU(3) , tout en considérant la réduction de SU(3) par rapport à son sous-groupe O(3) -et non par rapport au sous-groupe SU(2) [12] . Ainsi, le O(3) "orbital" mis en évidence dans le chapitre précédent (cas des mésons) serait "élargi" dans le cas des baryons en groupe SU(3) !

Ce travail est encore en cours d'étude, mais il va sans dire que l'introduction de ce nouvel SU(3) dans le cadre de la classification des hadrons mériterait quelque attention.

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CHAPITRE IV :

CURRENT QUARKS, CONSTITUENT QUARKS AND THE REACTIONS

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ABSTRACT : The transformation between current and constituent quarks is considered in view of the checking of the contributions, through P.C.A.C., of the Σ^*-resonances to the reactions $K^- p \to \pi^0 \Lambda(1405)$ and $K^- p \to \pi^0 \Lambda(1520)$, and the decays $\Sigma^* \to K^- p$ and $\Sigma^* \to \pi^0 \Lambda$. The Σ^*'s under consideration are classified in the 70,L = 1 representation together with the $\Lambda(1405)$ and $\Lambda(1520)$. In our model only few parameters are left, contrary to what happens in pure group theoretical approach.

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I - INTRODUCTION

The necessary distinction between the charge algebra and the group of classification for hadrons, picturized in the expressions "current quarks" and "constituent quarks"[1], suggested that a canonical transformation U, must exist which connects the two different algebras[2]. The first attempts to construct such a mixing operator show the viability and the predictive power of such an approach[3]. Since them, Melosh[4] has stressed what one can call the kinematical origin[4,5] of this transformation U, and has derived an expression for it in the framework of the free-quark model.

The revived interest on this subject has motivated many recent publications on the phenomenological applications[3,4,6,7,8]. Chiefly, this idea allowed for the construction of a simple and powerful model of axial charges. By the use of P.C.A.C., one can relate the resulting matrix elements of the axial charges between one hadron states to the pionic couplings of such hadrons.

The main tool has been to abstract from the free-quark model, the transformation properties of the axial charges under the collinear classification groups $SU(3) \times SU(3) \times O(2)$ and $SU(6)_W \times O(2)$ [6,7]: the general agreement with experiment is satisfactory and one reaches an interpretation of previous $SU(6)_W$ schemes[9] for the decays of resonances. However, this purely group-theoretical method does not spread any light on the reduced matrix elements which have to multiply the appropriate Clebsch-Gordan coefficients in order to get the desired axial couplings.

Further information can be reached by taking account of the way quarks appear to combine and build up baryon and meson states. In this framework, a more predictive approach has been developped[3,10], starting from the study of the saturation of the chiral algebra within a

realistic infinite set of states. It has lead to propose the following form for the axial charge[10]:

$$Q_5^\alpha = \sum_{\text{quarks}} \left\{ \cos\theta\left(\frac{k_\perp}{m}\right) \frac{\sigma_z \cdot \lambda^\alpha}{2} + \sin\theta\left(\frac{k_\perp}{m}\right) \frac{\sigma_x k_x + \sigma_y k_y}{2 k_\perp} \lambda^\alpha \right\}$$

Besides transforming as a 35 operator under $SU(6)_W$, this expression enjoys the property of obeying the selection rule[11] $\Delta(L - L_z)$ = even, which appears to be rather well verified[7,12] for both the hadron decay widths and the sign on the coupling, that are important in the inelastic reactions, like $\pi N \to \pi \Delta$ [13] and $\bar{K} p \to \pi \Sigma(1385)$ [14].

In this paper we consider the predictions of this model for the decays $\Sigma^* \to \Lambda(1405)\pi$, $\Sigma^* \to \Lambda(1520)\pi$, and for the contributions of the Σ^* resonances to the reactions $K^- p \to \Lambda(1405)\pi^0$ and $K^- p \to \Lambda(1520)\pi^0$, and we concentrate on the Σ^*'s which are classified in the 70(L=1) representation, where the $\Lambda(1405)$ and $\Lambda(1520)$ are also classified. We shall show that our model is quite predictive for the study of meson couplings to 70(L=1) baryons, unlike the purely group-theoretical approach, where too many reduced matrix elements are left as free parameters.

We have considered the above reactions because they are easier to study from the experimental point of view, particularly in the case of the $\Lambda(1520)$, which is a very narrow resonance. But the calculations can be immediately extended to any coupling between two 70(L=1) baryons, with no new parameters.

The paper is organized as follows :
Section II contains the group-theoretical approach to the baryon couplings[6,7], with the most general parametrization of the results, which turns out to be very poor of predictions. Our model is then introduced in Section III and only a few free parameters remain. It is applied to the calculation of decays and to the resonance contributions to the $K^- p \to \pi^0 \Lambda(1405)$, $\pi^0 \Lambda(1520)$ amplitudes, in Section IV.

II - THE ALGEBRAIC MODEL

Consider the residue of a Σ^*-resonance pole contributing to the forward $K^- p \to \pi^0 \Lambda^*$ amplitude, in the zero-width approximation ($h \equiv$ baryon helicity),

$$\text{Res}\, T_{hh}(s, \theta = 0) = \langle \pi^0 \Lambda^* | T_h | \Sigma_i^{0*} \rangle \langle \Sigma_i^{0*} | T_h | K^- p \rangle \qquad (2.1)$$

By PCAC, the on-shell π^0-coupling constants can be related to the matrix elements of the axial charges[15]:

$$\langle \pi^0 \Lambda^* | T_h | \Sigma_i^* \rangle = \frac{(m^2_{\Sigma_i^*} - m^2_{\Lambda^*})}{f_\pi} \langle \Lambda^* | Q_5^{\pi^0} | \Sigma_i^* \rangle_h \qquad (2.2a)$$

where $Q_5^{\pi^0}$ is the axial charge with the π^0 quantum-numbers, $f_\pi \approx 95$ Mev is the π-decay constant. Eq.(2.2a) can be extended to the K^- coupling constants if one assumes SU(3) symmetry for the meson-baryon couplings :

$$\langle \Sigma_i^* | T_h | K^- p \rangle = \frac{(m^2_{\Sigma_i^*} - m^2_p)}{f_\pi} \langle \Sigma_i^* | Q_5^{K^-} | p \rangle_h \qquad (2.2b)$$

Our next hypothesis is to take $SU(6)_S \times O(3)_L$ as the classification group for collinear hadron states : baryons are assumed to belong to representations of $SU(6)_S \times O(3)_L$, with the baryon spin being $\vec{J} = \vec{L} + \vec{S}$, and further, we suppose the SU(6) representation to be non exotic, corresponding to 3-quark states. We then write for the baryon states,

$$| \rho ; \mu, \nu ; L S ; J h \rangle = \sum_\lambda \begin{pmatrix} L & S \\ h-\lambda & \lambda \end{pmatrix} \Big| J \\ h \Big) \; | \rho, \mu, \nu, S, \lambda \rangle \, | L \; h-\lambda \rangle \qquad (2.3)$$

where : ρ is the $SU(6)_S$ representation ($\rho =$ 56, 70 or 20) ; μ and ν are SU(3) indices ($\mu =$ 1, 8 or 10 and $\nu = I$, I_z, Y) ; S and λ are the "total quark spin" eigenvalues ($S = \frac{3}{2}$ or $\frac{1}{2}$, λ being the S_3 eigenvalue) ; L is the "orbital angular momentum" and J and h are the baryon spin and helicity (the z-direction is taken along the collinear motion).

Now there are two inequivalent 70 representations as long as the symmetry of the states under permutation of the quarks are considered. For further use, we shall write the physical states as

$$|70\rangle|L\rangle = \frac{1}{\sqrt{2}}\left\{ a_\alpha |70\rangle_\alpha |L\rangle_\alpha + a_\beta |70\rangle_\beta |L\rangle_\beta \right\} \qquad (2.4)$$

The 70_α and 70_β states are two independent 70 states to be chosen in a convenient way, and $a_\alpha^2 + a_\beta^2 = 2$. The states $|L\rangle_\alpha$ and $|L\rangle_\beta$ differ by some quantum number other than L in general.

Following Refs.(1-9), we assume that the axial charges are related to the generators of the $SU(6)_S$ group by the transformation U. They are then supposed to have the following structure,

$$Q_S^\alpha = C X^\alpha + S_+ Y_-^\alpha + S_- Y_+^\alpha \qquad (2.5)$$

The operators X^α and $Y_\pm^\alpha$ are O(3) invariant, behave as 35 operators under $SU(6)_S$, as $SU(3)$ octets and as $SU(2)_S$ vectors, with $S_z = 0$ for X^α and $S_z = \pm 1$ for $Y_\pm$ (*). Instead $C, S_\pm$ are SU(6) scalars, are highly reducible in $O(3)_L$ but $L_z = 0$ for C and $L_z = \pm 1$ for $S_\pm$. Many consequences of these assumptions have been successfully tested(3-9).

Now that both the states and the Q_S^α operators are classified under $SU(6) \times O(3)$, one can proceed to apply the Wigner-Eckart theorem to the matrix elements of the axial charge between baryon states and write :

$$\langle \rho' \mu' \nu' L' S' J' h | Q_S^\alpha | \rho \mu \nu L S J h \rangle = G(\rho L S J h, \rho' L' S' J' h) \times \qquad (2.6)$$
$$\times \langle \rho' \mu' \nu' S' \lambda = \tfrac{1}{2} | X^\alpha | \rho \mu \nu S \lambda = \tfrac{1}{2} \rangle$$

(*) The distinction between W-spin and S-spin being unimportant for baryons.

where :

$$\langle \rho' \mu' \nu' S' \lambda = \tfrac{1}{2} | X^{\alpha} | \rho \mu \nu S \lambda = \tfrac{1}{2} \rangle = \sum_{\gamma} \begin{pmatrix} \rho & & \underline{35} & \rho' \\ \mu, \nu, S, \lambda = \frac{1}{2} & & \underline{8}, \alpha, S=1, \lambda=0 & \mu'_{\gamma}, \nu', S', \lambda' = \frac{1}{2} \end{pmatrix} \times A(\rho, \rho') \tag{2.7}$$

is an SU(6) Clebsh-Gordan coefficient (the index γ labelling the two independent octets, $\mu'_{\gamma} = 8_a$ or 8_s, when $\mu = 8$), with an arbitrary normalization, represented by the factor $A(\rho, \rho')$. The spin and angular momentum dependence, together with the unknown reduced matrix elements are contained in,

$$G(\rho L S J h, \rho' L' S' J' h) = \tag{2.8}$$

$$\sum_{\lambda \lambda'} \begin{pmatrix} L' & S' & J \\ h-\lambda' & \lambda' & h \end{pmatrix} \begin{pmatrix} L & S & J \\ h-\lambda & \lambda & h \end{pmatrix} \left[\begin{pmatrix} S & 1 & S' \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \right]^{-1} A(\rho, \rho')^{-1}$$

$$\times \left\{ \begin{pmatrix} S & 1 & S' \\ \lambda & 0 & \lambda' \end{pmatrix} \langle L' \; h-\lambda' | C | L \; h-\lambda \rangle \langle \rho' \| X \| \rho \rangle \right.$$

$$+ \begin{pmatrix} S & 1 & S' \\ \lambda & +1 & \lambda' \end{pmatrix} \langle L' \; h-\lambda' | S_- | L \; h-\lambda \rangle \langle \rho' \| Y \| \rho \rangle$$

$$\left. + \begin{pmatrix} S & 1 & S' \\ \lambda & -1 & \lambda' \end{pmatrix} \langle L' \; h-\lambda | S_+ | L \; h-\lambda \rangle \langle \rho' \| Y \| \rho \rangle \right\}$$

Now we can consider the particular matrix elements we are interested in. The corresponding SU(6) Clebsh-Gordan coefficients are given in Table I. The G factors for $L' = 1$ and $L = 0, 1$ are expressed in Table II in terms of the following parameters :

$$\gamma = \langle L=1, L_z=0 | C | L=0 \rangle \; \langle 70 \| X \| 56 \rangle$$

$$\sigma = \langle L=1, L_z=\pm 1 | S_{\pm} | L=0 \rangle \; \langle 70 \| Y \| 56 \rangle \tag{2.9}$$

$$\delta^{\alpha\beta}_{m} = \langle L=1, L_z=m | C | L=1, L_z=m \rangle_{\alpha\beta} \; \langle 70 \| X \| 70 \rangle_{\alpha\beta} \; a^{\varepsilon}_{\alpha\beta}$$

$$c_{\pm}^{\alpha,\beta} = \sqrt{2}\,\langle L=1,\ L_z=\pm 1 |\ S_{\pm}\ |\ L=1,\ L_z=0\rangle_{\alpha,\beta}\ \langle 70 \| Y \| 70\rangle_{\alpha,\beta}\ a^{\ell}_{\alpha,\beta}$$

$$= -\sqrt{2}\,\langle L=1,\ L_z=0 |\ S_{\mp}\ |\ L=1,\ L_z=\pm 1\rangle_{\alpha,\beta}\ \langle 70 \| Y \| 70\rangle_{\alpha,\beta}\ a^{\ell}_{\alpha,\beta}$$

The α and β-states were chosen without any loss of generality in such a way that the charges are diagonal in these indices.

From these Tables one can express the matrix elements in Eq.(2.2) as linear combinations of these parameters.

Unfortunately, there are too many parameters in this model to derive any prediction. As we shall see, the model discussed in the next section allows for the determination of many of the parameters in Eq.(2.9) and gives a very predictive framework. In this sense, it is a phenomenological explicit realization of the algebraic model.

III - A SYMMETRIC QUARK MODEL

We shall consider the baryons as 3-constituent quark systems $(q_1 q_2 q_3)$, each quark having momentum $\vec{k}^i$ $(i=1,2,3)$ in the C.M. of the baryon, i.e. $\sum_i \vec{k}_i = 0$. This C.M. momentum can be given a covariant definition[16]. We introduce two independent internal momenta

$$\vec{k}^\alpha = \frac{1}{\sqrt{6}}\left[2\vec{k}_1 - \vec{k}_2 + \vec{k}_3\right] \qquad \vec{k}^\beta = \frac{1}{\sqrt{2}}\left[\vec{k}_2 - \vec{k}_3\right] \tag{3.1}$$

and, correspondingly, two independent internal angular momentum operators,

$$\vec{L}^\alpha = -i\,\vec{k}^\alpha \times \frac{\partial}{\partial \vec{k}^\alpha} \qquad \vec{L}^\beta = -i\,\vec{k}^\beta \times \frac{\partial}{\partial \vec{k}^\beta}$$
$$\vec{L} = \vec{L}^\alpha + \vec{L}^\beta \tag{3.2}$$

where $\vec{L}$ is the total orbital angular momentum.

Then, the baryon states which are classified in irreducible representations of $SU(6)\times O(3)$ can be written as follows :

$$|\rho;\mu,\nu;S,\lambda\rangle\,|L L_z\rangle = \sum_{q_1 q_2 q_3} \int d\vec{k}_1\, d\vec{k}_2\, d\vec{k}_3\, \delta(\vec{k}_1+\vec{k}_2+\vec{k}_3)\times$$
$$\times\, B^{\rho\mu\nu S\lambda}_{q_1 q_2 q_3}\; B_{LL_z}(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,|\hat{q}_1,\vec{k}^1\rangle\,|\hat{q}_2,\vec{k}^2\rangle\,|\hat{q}_3,\vec{k}^3\rangle \tag{3.3}$$

where $B^{\rho\mu\nu S\lambda}_{q_1 q_2 q_3}$ are the Clebsh-Gordan coefficients for reduction of the 6x6x6 representation of SU(6) and $B_{LL_z}(\vec{k}^i)$ are the spatial wave functions. The baryon spin is $\vec{J} = \vec{L} + \sum_i \vec{S}^i$ where $\vec{S}^i$ is the C.M. spin operator for the i-th (constituent) quark[5].

The mixing problem arises because of the C.M. definition of the spin for the constituent quarks : in order to have current-quarks, whose spin is defined in the quark rest system, we must introduce the

transformation betwwen constituent and current quarks, which, even in the presence of interactions, should be the analogous of the Wigner rotation of spins arising from the Lorentz transformation between the quark rest frame and the C.M. frame. By analogy with the free-quark case we write[10]:

$$|\hat{q}_i, \vec{k}^i\rangle = U_i(\vec{k}^i)\ |q_i, \vec{k}^i\rangle$$

$$U_i(\vec{k}^i) = \exp\left[i\,\theta(k_\perp^i, x^i)\ \frac{(\vec{\sigma}^i \times \vec{k}^i)_z}{k_\perp^i}\right] \qquad (3.4)$$

$$\vec{k}_\perp^i = (k_{1i}, k_2^i, 0) \quad ; \quad x^i = k_0^i + k_3^i$$

One can now assume that the axial charges are simply written for the three current quark states as

$$\langle q_1' q_2' q_3' \mid Q_5^\alpha \mid q_1 q_2 q_3\rangle = \langle q_1' q_2' q_3' \mid \sum_i \frac{\lambda_\alpha^i \sigma_z^i}{2} \mid q_1 q_2 q_3\rangle \qquad (3.5)$$

(the operators with an index i act only on the i-th quark). Therefore, for three constituent quark states one has the following realization of the axial charges[10]:

$$Q_5^\alpha = \left[\prod_i U_i^{-1}(\vec{k}^i)\right]\left[\sum_i \frac{\lambda_\alpha^i \cdot \sigma_z^i}{2}\right]\left[\prod_i U_i(\vec{k}^i)\right]$$

$$= \sum_i \left\{\cos\theta_i(\vec{k}^i)\ \frac{\lambda_\alpha^i \cdot \sigma_z^i}{2} + \sin\theta_i(\vec{k}^i)\ \lambda_\alpha^i\ \frac{\vec{\sigma}^i \cdot \vec{k}_\perp^i}{2 k_\perp^i}\right\} \qquad (3.6)$$

This expression is an explicit realization of the general algebraic structure discussed for Eq.(2.5). The advantage of such a model is that it allows for the calculation of the reduced matrix elements which were left as parameters in the previous formalism, once one has an explicit form for the baryon wave-functions.

One can incorporate into the model the phenomenological assumptions of the "even $(\Delta L - \Delta L_z)$ rule"[3,11], by assuming that

$\theta_i\left(k_\perp^i, x^i\right) = \theta_i\left(k_\perp^i\right)$ so that this operator will contain only even values of L . In this way the axial charge will decompose into an even L , $L_z = 0$ piece and another odd L, $L_z = \pm 1$ one, as required by the above rule.

We shall also assume the symmetric quark model, where the baryon $SU(6) \times O(3)$ wave-function is supposed to be symmetric in the 3 quarks. We follow the formalism of ref.(17) and write for the symmetric baryon states :

$$|56\rangle\,|L\rangle_S \qquad\qquad |20\rangle\,|L\rangle_A$$

and $$\frac{1}{\sqrt{2}}\left(|70\rangle_\alpha\,|L\rangle_\alpha + |70\rangle_\beta\,|L\rangle_\beta\right)$$

where the subscripts give the permutational symmetry of the spatial configuration : S for symmetric, A for antisymmetric, α and β for mixed symmetric states.

Therefore, any dynamical operator must act symmetrically in the 3 quarks in order to connect only physical (symmetric) states. The axial charge in Eq.(3.6) has already this property, so that one simplify the calculations by making use of the identity

$$\langle B'|\,Q_5^q\,|B\rangle =$$

$$3\,\langle B'|\left\{\cos\theta_1\left(\vec{k}'\right)\frac{\lambda_\alpha'^1\cdot\sigma_2'^1}{2} + \sin\theta_1\left(\vec{k}'\right)\frac{\lambda_\alpha'^1\cdot\vec{\sigma}'^1\cdot\vec{k}_\perp'^1}{2\,k_\perp'^1}\right\}|B\rangle$$

for any baryons B,B'.

With these definitions for the axial charges and baryon wave-functions, one gets the same results as in Table I and Table II (the algebraic structure of Eqs.(3.7) and (2.5) are the same) but now the parameters can be explicitely given as

$$\gamma^{\alpha_1\beta} = 3 \underset{\alpha_1\beta}{\langle} L=1, L_z=0 \,|\, \cos\theta_1(\vec{k}^1) \,|\, L=0 \rangle$$

$$\sigma^{\alpha_1\beta} = 3 \underset{\alpha_1\beta}{\langle} L=1, L_z=\pm 1 \,|\, \frac{k^1_\pm}{k^1_\perp} \sin\theta(\vec{k}^1) \,|\, L=0 \rangle$$

$$\delta_m^{\alpha_1\beta} = 3 \underset{\alpha_1\beta}{\langle} L=1, L_z=m \,|\, \cos\theta_1(\vec{k}^1) \,|\, L=1, L_z=m \rangle_{\alpha_1\beta} \qquad (3.8)$$

$$\tau_\pm^{\alpha_1\beta} = 3\sqrt{2} \underset{\alpha_1\beta}{\langle} L=1, L_z=\pm 1 \,|\, \frac{k^1_\pm}{k^1_\perp} \sin\theta(\vec{k}^1) \,|\, L=1, L_z=0 \rangle_{\alpha_1\beta}$$

Now one can consider the separation into two internal motions $(\alpha_1\ \beta)$ and one sees from Eqs. (3.1), (3.2) that $\vec{L}^\beta$ and $\vec{k}^1$ commute. Therefore

$$\overset{\beta}{m} = \langle L^\alpha=0 \,|\, \cos\theta(\vec{k}^1) \,|\, L^\alpha=0 \rangle \langle L^\beta=1, L^\beta_z=m \,|\, L^\beta=1, L^\beta_z=m \rangle \qquad (3\ 9a)$$

$$= \langle L=0 \,|\, \cos\theta(\vec{k}^1) \,|\, L=0 \rangle = \frac{3}{5} G_A \simeq \frac{3}{4}$$

where the last equality follows from the calculations for the matrix elements of Q_5 between nuleon states[3,4,6].

Analogously, one gets

$$\tau_\pm^\beta = \gamma^\beta = \sigma^\beta = 0 \qquad (3\ 9b)$$

If now the "even $(\Delta L - \Delta L_z)$ rule" (or $\theta_i(k^i_\perp, x^i) = \theta_i(k^i_\perp)$) is assumed, one has

$$\gamma^\alpha = 0 \qquad \delta^\alpha_+ = \delta^\alpha_- \qquad \delta^\alpha_0 \gg \delta^\alpha_+ \gg \frac{\delta^\alpha_0}{2} \qquad \tau^\alpha_+ = \tau^\alpha_- \qquad (3.10)$$

In order to get more restrictions on these parameters, one should know something about the "spatial" wave-functions. If a harmonic oscillator spectrum is assumed, as often done in the quark model, one gets for these $n = 1$ matrix elements(x)

(x) Even without reference to the harmonic oscillator one could expected Eq. (3.11) to be approximately satisfied from more general considerations, as only the $L = 2$ part of the operator

$$\frac{\sin\theta(k^1_\perp)}{k^1_\perp}$$ can contribute to these matrice elements.

$$c_+^\alpha = c_-^\alpha = 0 \tag{3.11}$$

$$\delta_0^\alpha = \delta^\beta \tag{3.12}$$

Therefore, in the explicit model considered here, the number of parameters is very reduced. Then one can write the matrix elements which appear in Eq.(2.2) in terms of them. The results are given in Table III.

IV - RESULTS

Now, we can apply this phenomenological model to the Σ^* decays. From Eq.(2.2) one deduces the following expression[10] for the decay $\Sigma^* \to \Lambda^* + \pi$:

$$\Gamma\left(\Sigma^* \to \Lambda^* \pi\right) = \left[\begin{matrix}\text{CORRECTION}\\ \text{FACTOR}\end{matrix}\right] \frac{\left(m^2_{\Sigma^*} - m^2_{\Lambda^*}\right)^3}{16\pi\, \rho^2_\pi\, m^3_{\Sigma^*}\left(2J_{\Sigma^*}+1\right)} \sum_h \langle \Lambda^* | Q_S^{\pi^0} | \Sigma^* \rangle_h \qquad (4.1)$$

The correction factor takes into account the non vanishing mass of the meson and the expression suggested by Weinberg[10] for this factor is

$$\left[\begin{matrix}\text{CORRECTION}\\ \text{FACTOR}\end{matrix}\right] \simeq \left[1 - \frac{2\mu^2\left(m^2_{\Sigma^*} + m^2_{\Lambda^*}\right)}{\left(m^2_{\Sigma^*} - m^2_{\Lambda^*}\right)^2}\right]^{\frac{2l+1}{2}}$$

where l is the orbital angular momentum of the $\Lambda^* \pi$ system. In this model, all the decays $\Sigma^*_{L=1} \to \Lambda^*_{L=1} \pi$ are predicted to be mostly p-wave $(l=1)$.

Analogously, one obtains the expressions for the decays $\Sigma^{*0} \to p \pi^-$ from Eq.(4.1) by replacing $\Lambda^* \to p$ and $\pi^0 \to K^-$.

By using the matrix elements given in Table III one can calculate the decay widths presented in Tables IV and V We assume the following classification of the particles we are going to consider :

$$|p\rangle_h = |\underline{56}\,;\, P=8\,;\, S=\tfrac{1}{2}\,,\, L=0\,;\, J=\tfrac{1}{2}\,,\, h\rangle$$

$$|\Lambda(1405)\rangle_h = |\underline{70}\,;\, P=1\,,\, S=\tfrac{1}{2}\,,\, L=1\,,\, J=\tfrac{1}{2}\,,\, h\rangle$$

$$|\Lambda(1520)\rangle_h = |\underline{70}\,;\, P=1\,;\, S=\tfrac{1}{2}\,,\, L=1\,,\, J=\tfrac{3}{2}\,,\, h\rangle$$

The identification of the $J^P = \frac{3}{2}^-$ and $\frac{1}{2}^-$ Σ^* states is tentative : we take the experimentally observed states as those

predicted to be more coupled to $\bar{K}p$, which turn out to be the $S = \frac{3}{2}$ ones. As a remarkable consequence, only the δ^β parameter appears in the $\Sigma^* \Lambda^* \pi$ couplings, and because of the result in Eq.(3.9a) one can calculate these coupling constants without free parameters. The results are quite encouraging. The results reported in Table V coincide with those obtained in Ref.(3), and are quite satisfactory (the value $\rho^2 \approx \frac{1}{3}$ is taken from the $N^*(1520) \to N\pi$ decay).

One can also calculate the signs (and magnitudes) of the $\bar{K}p \to \Lambda^* \pi^0$ amplitudes at the resonance energies. The results are given in Table VI. Even if the parameters δ_+^α, δ_0^α are not known, they can be estimated from the inequality in Eq.(3.10) and by assuming Eq.(3.12) to be roughly satisfied. Then, all the relative signs can be determined, but one. They are given in the last column of Table VI.

V - CONCLUDING REMARKS

It is clear that most of these parameter-free predictions are characteristic of the model in Sect. III and cannot be obtained in the more general formalism of Sect. II . Therefore the experimental determination of the physical quantities calculated in this paper should provide a direct test for our simple and explicit model for the axial charges.

A more detailed calculation should take into account the representation mixing which occur in the SU(6) classification. The phenomenological fits[19] give relatively small mixing angles for the 70(L=1) baryons (unlike the $\omega - \phi$ case) and, presumably, this technical complication should not spoil the main features of our results. The inclusion of mixing angles is in study now.

Analogolous calculations can be done both by considering the contributions of 56(L=2) baryons and by looking at other production amplitudes.

It is worthnoticing that the satisfactory agreement of the widths calculated here with the experimental results give some indication on the behaviour of the quarks inside the baryons.

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COEFFICIENTS $\langle \rho', \gamma', \nu', S', \lambda = \frac{1}{2} \mid X^{\pi^0} \mid \rho, \gamma, \nu, S, \lambda = \frac{1}{2} \rangle$ (See Eq. (2.7)

$\rho\ \gamma\ \nu$	56, 10, Σ^0, $\frac{3}{2}$	56, 8, Σ^0, $\frac{1}{2}$	70, 8, Σ^0, $\frac{3}{2}$		70, 8, Σ^0, $\frac{1}{2}$		70, 10, Σ^0, $\frac{1}{2}$		2a, 8, Σ^0, $\frac{1}{2}$
			λ	β	α	β	α	β	
56, 8, A, $\frac{1}{2}$	$\sqrt{\frac{2}{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$		$-\frac{1}{2\sqrt{3}}$		$\frac{1}{2\sqrt{3}}$		C
70, 8, A, $\frac{3}{2}$	$-\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{2}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$
70, 8, A, $\frac{1}{2}$	$\sqrt{\frac{2}{3}}$	$-\frac{1}{2\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$-\frac{1}{\sqrt{3}}$	$\frac{1}{2\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2\sqrt{3}}$
70, 1, A, $\frac{1}{2}$	0	$-\frac{\sqrt{3}}{2}$	0	$-\frac{2}{\sqrt{3}}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2\sqrt{3}}$	0	0	$-\frac{1}{2}$
2a, 8, A, $\frac{1}{2}$	0	0	$-\frac{1}{\sqrt{3}}$		$-\frac{1}{2\sqrt{3}}$		$\frac{\sqrt{3}}{2}$		$-\frac{1}{\sqrt{3}}$
2a, 10, A, $\frac{1}{2}$	0	0	$\frac{1}{\sqrt{6}}$		$\sqrt{\frac{2}{3}}$		0		$-\sqrt{\frac{2}{3}}$

TABLE I

$G_{\uparrow}\left(J,\ L=1,\ S\ ;\ J',\ L'=1,\ S'\right)$

$J,\ S^{\uparrow}$	$\frac{5}{2},\ \frac{3}{2}$	$\frac{3}{2},\ \frac{3}{2}$	$\frac{1}{2},\ \frac{3}{2}$	$\frac{3}{2},\ \frac{1}{2}$	$\frac{1}{2},\ \frac{1}{2}$
$\frac{5}{2},\ \frac{3}{2}$	$\frac{3}{5}\{\delta_0 - \frac{1}{2}\delta_+ + \frac{1}{2}\delta_- + 2\tau_+ - \tau_-\}$	$-\frac{1}{5}\{\delta_0 + 2\delta_+ + 3\delta_- - 3\tau_+ - \frac{7}{2}\tau_-\}$	$-\frac{1}{\sqrt{5}}\{\delta_0 + \frac{1}{2}\delta_+ - \frac{3}{2}\delta_- - \tau_-\}$	$\sqrt{\frac{2}{5}}\{\delta_0 + \frac{1}{2}\delta_+ + \frac{1}{4}\tau_-\}$	$-\frac{1}{\sqrt{5}}\{\delta_0 - \delta_+ + \frac{3}{4}\tau_+ + \frac{1}{4}\tau_-\}$
$\frac{3}{2},\ \frac{3}{2}$	$-\frac{1}{5}\{\delta_0 + 2\delta_+ + 3\delta_- - 3\tau_+ - \frac{7}{2}\tau_-\}$	$\frac{1}{15}\{\delta_0 - 8\delta_+ + 6\delta_- - 8\tau_+ - 2\tau_-\}$	$\frac{1}{3\sqrt{5}}\{\delta_0 - 2\delta_+ - 9\delta_- - 5\tau_+ - \frac{3}{2}\tau_-\}$	$-\frac{\sqrt{2}}{3\sqrt{5}}\{\delta_0 - 2\delta_+ - \frac{5}{4}\tau_+ + \frac{3}{2}\tau_-\}$	$\frac{1}{3\sqrt{5}}\{\delta_0 + 4\delta_+ - \frac{1}{2}\tau_+ + \frac{3}{2}\tau_-\}$
$\frac{1}{2},\ \frac{3}{2}$	$-\frac{1}{\sqrt{5}}\{\delta_0 + \frac{1}{2}\delta_+ - \frac{3}{2}\delta_- - \tau_-\}$	$\frac{1}{3\sqrt{5}}\{\delta_0 - 2\delta_+ - 9\delta_- - 5\tau_+ - \frac{3}{2}\tau_-\}$	$\frac{1}{3}\{\delta_0 - \frac{1}{2}\delta_+ + \frac{9}{2}\delta_- - 2\tau_+ + 3\tau_-\}$	$-\frac{\sqrt{2}}{3}\{\delta_0 - \frac{1}{2}\delta_+ - \frac{1}{2}\tau_+ - \frac{3}{4}\tau_-\}$	$\frac{1}{3}\{\delta_0 + \delta_+ + \frac{1}{4}\tau_+ - \frac{3}{4}\tau_-\}$
$\frac{3}{2},\ \frac{1}{2}$	$\sqrt{\frac{2}{5}}\{\delta_0 + \frac{1}{2}\delta_+ + \frac{1}{2}\tau_-\}$	$-\frac{\sqrt{2}}{3\sqrt{5}}\{\delta_0 - 2\delta_+ - \frac{5}{4}\tau_+ + \frac{3}{2}\tau_-\}$	$-\frac{\sqrt{2}}{3}\{\delta_0 - \frac{1}{2}\delta_+ - \frac{1}{2}\tau_+ - \frac{3}{4}\tau_-\}$	$\frac{2}{3}\{\delta_0 - \frac{1}{2}\delta_+ + \tau_+\}$	$-\frac{\sqrt{2}}{3}\{\delta_0 + \delta_+ - \frac{1}{2}\tau_+\}$
$\frac{1}{2},\ \frac{1}{2}$	$-\frac{1}{\sqrt{5}}\{\delta_0 - \delta_+ + \frac{3}{4}\tau_+ + \frac{1}{4}\tau_-\}$	$\frac{1}{3\sqrt{5}}\{\delta_0 + 4\delta_+ - \frac{1}{2}\tau_+ + \frac{3}{2}\tau_-\}$	$\frac{1}{3}\{\delta_0 + \delta_+ + \frac{1}{4}\tau_+ - \frac{3}{4}\tau_-\}$	$-\frac{\sqrt{2}}{3}\{\delta_0 + \delta_+ - \frac{1}{2}\tau_+\}$	$\frac{1}{3}\{\delta_0 - 2\delta_+ - 2\tau_+\}$

TABLE II

$\langle B' | Q_5^{\pi^0} | B \rangle_{h=+\frac{1}{2}}$

B' \ B	$\Lambda^{(1)}$ $J=\frac{1}{2}$, $S=\frac{1}{2}$	$\Lambda^{(1)}$ $J=\frac{3}{2}$, $S=\frac{1}{2}$	$\Lambda^{(8)}$ $J=\frac{1}{2}$, $S=\frac{1}{2}$	$\Lambda^{(8)}$ $J=\frac{3}{2}$, $S=\frac{1}{2}$
$\Sigma^{(8)}_{3/2}\left(\frac{5}{2}^-\right)$	0	$-\sqrt{\frac{3}{10}}\,\delta^{\beta}$	$-\frac{1}{2\sqrt{15}}\delta_0^{\alpha}+\frac{1}{2\sqrt{15}}\delta_+^{\alpha}$	$\frac{1}{\sqrt{30}}\delta_0^{\alpha}+\frac{1}{2\sqrt{30}}\delta_+^{\alpha}+\frac{3}{2\sqrt{30}}\delta^{\beta}$
$\Sigma^{(8)}_{3/2}\left(\frac{3}{2}^-\right)$	$-\frac{\sqrt{5}}{3\sqrt{3}}\delta^{\beta}$	$-\frac{2}{3\sqrt{30}}\delta^{\beta}$	$\frac{1}{6\sqrt{15}}\delta_0^{\alpha}+\frac{2}{3\sqrt{15}}\delta_+^{\alpha}+\frac{\sqrt{5}}{6\sqrt{3}}\delta^{\beta}$	$-\frac{1}{3\sqrt{30}}\delta_0^{\alpha}+\frac{\sqrt{2}}{3\sqrt{15}}\delta_+^{\alpha}+\frac{1}{3\sqrt{30}}\delta^{\beta}$
$\Sigma^{(8)}_{\frac{1}{2}}\left(\frac{3}{2}^-\right)$	$-\frac{2}{3\sqrt{3}}\delta^{\beta}$	$\frac{1}{3\sqrt{6}}\delta^{\beta}$	$\frac{1}{6\sqrt{3}}\delta_0^{\alpha}+\frac{1}{6\sqrt{3}}\delta_+^{\alpha}+\frac{1}{3\sqrt{3}}\delta^{\beta}$	$-\frac{1}{3\sqrt{6}}\delta_0^{\alpha}+\frac{1}{6\sqrt{6}}\delta_+^{\alpha}-\frac{1}{6\sqrt{6}}\delta^{\beta}$
$\Sigma^{(8)}_{\frac{1}{2}}\left(\frac{3}{2}^-\right)$	$-\frac{1}{3\sqrt{6}}\delta^{\beta}-\frac{1}{2\sqrt{6}}\delta_0^{\alpha}-\frac{1}{2\sqrt{6}}\delta_+^{\alpha}$	$\frac{1}{12\sqrt{3}}\delta^{\beta}+\frac{1}{2\sqrt{3}}\delta_0^{\alpha}-\frac{1}{4\sqrt{3}}\delta_+^{\alpha}$	$-\frac{1}{3\sqrt{6}}\delta_0^{\alpha}-\frac{1}{3\sqrt{6}}\delta_+^{\alpha}+\frac{\sqrt{2}}{3\sqrt{3}}\delta^{\beta}$	$\frac{1}{3\sqrt{3}}\delta_0^{\alpha}-\frac{1}{6\sqrt{3}}\delta_+^{\alpha}-\frac{1}{6\sqrt{3}}\delta^{\beta}$
$\Sigma^{(8)}_{\frac{1}{2}}\left(\frac{1}{2}^-\right)$	$-\frac{1}{12\sqrt{3}}\delta^{\beta}+\frac{1}{4\sqrt{3}}\delta_0^{\alpha}-\frac{1}{2\sqrt{3}}\delta_+^{\alpha}$	$-\frac{1}{3\sqrt{6}}\delta^{\beta}-\frac{1}{2\sqrt{6}}\delta_0^{\alpha}-\frac{1}{2\sqrt{6}}\delta_+^{\alpha}$	$\frac{1}{6\sqrt{3}}\delta_0^{\alpha}-\frac{1}{3\sqrt{3}}\delta_+^{\alpha}+\frac{1}{6\sqrt{3}}\delta^{\beta}$	$-\frac{1}{3\sqrt{6}}\delta_0^{\alpha}-\frac{1}{3\sqrt{6}}\delta_+^{\alpha}+\frac{2}{3\sqrt{6}}\delta^{\beta}$
$\Sigma^{(10)}_{\frac{1}{2}}\left(\frac{3}{2}^-\right)$	0	0	$-\frac{1}{6\sqrt{6}}\delta_0^{\alpha}-\frac{1}{6\sqrt{6}}\delta_+^{\alpha}-\frac{1}{\sqrt{6}}\delta^{\beta}$	$\frac{1}{6\sqrt{3}}\delta_0^{\alpha}-\frac{1}{12\sqrt{3}}\delta_+^{\alpha}+\frac{1}{4\sqrt{3}}\delta^{\beta}$
$\Sigma^{(10)}_{\frac{1}{2}}\left(\frac{1}{2}^-\right)$	0	0	$\frac{1}{12\sqrt{3}}\delta_0^{\alpha}-\frac{1}{6\sqrt{3}}\delta_+^{\alpha}-\frac{1}{4\sqrt{3}}\delta^{\beta}$	$-\frac{1}{2\sqrt{6}}\delta_0^{\alpha}-\frac{1}{6\sqrt{6}}\delta_+^{\alpha}-\frac{1}{\sqrt{6}}\delta^{\beta}$

TABLE II (Beginning)

$\langle B' | Q_5^{\pi^0} | B \rangle_{h=+\frac{1}{2}}$

$\langle B' | Q_5^{K^-} | p(\frac{1}{2}^+) \rangle_{h=+\frac{1}{2}}$

B' \ B	$\Lambda^{(8)}$ $J=\frac{1}{2}$, $S=\frac{3}{2}$	Λ^{8} $J=\frac{3}{2}$, $S=\frac{3}{2}$	$\Lambda^{(8)}$ $J=\frac{5}{2}$, $S=\frac{3}{2}$	p $J=\frac{1}{2}$, $S=\frac{1}{2}$
$\Sigma^{(8)}_{3/2}(\frac{5}{2}^-)$	$\frac{1}{4\sqrt{15}}\delta_0^\alpha - \frac{1}{4\sqrt{15}}\delta_+^\alpha$	$\frac{1}{20\sqrt{3}}\delta_0^\alpha + \frac{1}{4\sqrt{3}}\delta_+^\alpha - \frac{3}{10\sqrt{3}}\delta^\beta$	$-\frac{3}{20\sqrt{3}}\delta_0^\alpha + \frac{3}{20\sqrt{3}}\delta_0^\beta$	$-\frac{1}{\sqrt{15}}p$
$\Sigma^{(8)}_{\frac{3}{2}}(\frac{3}{2}^-)$	$-\frac{1}{12\sqrt{15}}\delta_0^\alpha + \frac{11}{12\sqrt{15}}\delta_+^\alpha - \frac{5}{6\sqrt{15}}\delta^\beta$	$-\frac{1}{60\sqrt{3}}\delta_0^\alpha + \frac{1}{30\sqrt{3}}\delta_+^\alpha + \frac{1}{60\sqrt{3}}\delta^\beta$	$\frac{1}{20\sqrt{3}}\delta_0^\alpha + \frac{1}{4\sqrt{3}}\delta_+^\alpha - \frac{3}{10\sqrt{3}}\delta^\beta$	$\frac{1}{3\sqrt{15}}p$
$\Sigma^{(8)}_{\frac{3}{2}}(\frac{1}{2}^-)$	$-\frac{1}{12\sqrt{3}}\delta_0^\alpha - \frac{1}{3\sqrt{3}}\delta_+^\alpha + \frac{5}{12\sqrt{3}}\delta^\beta$	$-\frac{1}{12\sqrt{15}}\delta_0^\alpha + \frac{11}{12\sqrt{15}}\delta_+^\alpha + \frac{5}{6\sqrt{15}}\delta^\beta$	$\frac{1}{4\sqrt{15}}\delta_0^\alpha - \frac{1}{4\sqrt{15}}\delta_+^\alpha$	$-\frac{2}{3\sqrt{3}}p$
$\Sigma^{(8)}_{\frac{1}{2}}(\frac{3}{2}^-)$	$-\frac{1}{3\sqrt{6}}\delta_0^\alpha + \frac{1}{6\sqrt{6}}\delta_+^\alpha - \frac{1}{6\sqrt{6}}\delta^\beta$	$-\frac{1}{3\sqrt{30}}\delta_0^\alpha + \frac{2}{3\sqrt{30}}\delta_+^\alpha + \frac{1}{3\sqrt{30}}\delta^\beta$	$\frac{1}{\sqrt{30}}\delta_0^\alpha + \frac{1}{2\sqrt{30}}\delta_+^\alpha + \frac{3}{2\sqrt{30}}\delta^\beta$	$-\frac{1}{6\sqrt{6}}p$
$\Sigma^{(8)}_{\frac{1}{2}}(\frac{1}{2}^-)$	$\frac{1}{6\sqrt{3}}\delta_0^\alpha + \frac{1}{6\sqrt{3}}\delta_+^\alpha + \frac{1}{3\sqrt{3}}\delta^\beta$	$\frac{1}{6\sqrt{15}}\delta_0^\alpha + \frac{2}{3\sqrt{15}}\delta_+^\alpha + \frac{5}{6\sqrt{15}}\delta^\beta$	$-\frac{1}{2\sqrt{15}}\delta_0^\alpha + \frac{1}{2\sqrt{15}}\delta_+^\alpha$	$-\frac{1}{6\sqrt{3}}p$
$\Sigma^{(10)}_{\frac{1}{2}}(\frac{3}{2}^-)$	$-\frac{\sqrt{2}}{3\sqrt{3}}\delta_0^\alpha + \frac{1}{3\sqrt{6}}\delta_+^\alpha$	$-\frac{\sqrt{2}}{3\sqrt{15}}\delta_0^\alpha + \frac{2\sqrt{2}}{3\sqrt{15}}\delta_+^\alpha$	$\frac{\sqrt{2}}{\sqrt{15}}\delta_0^\alpha + \frac{1}{\sqrt{30}}\delta_+^\alpha$	$\frac{1}{6\sqrt{6}}p$
$\Sigma^{(10)}_{\frac{1}{2}}(\frac{1}{2}^-)$	$\frac{1}{3\sqrt{3}}\delta_0^\alpha - \frac{1}{3\sqrt{3}}\delta_+^\alpha$	$\frac{1}{3\sqrt{15}}\delta_0^\alpha - \frac{4}{3\sqrt{15}}\delta_+^\alpha$	$-\frac{1}{\sqrt{15}}\delta_0^\alpha + \frac{1}{\sqrt{15}}\delta_+^\alpha$	$\frac{1}{6\sqrt{3}}p$

TABLE III (end)

DECAYS $\Sigma^0(10, L=1) \longrightarrow p K^-$

$\Sigma_{(L=1)}$	MASS	Kinem. Factor (Mev.)	$\sum_h \lvert\langle \Sigma \lvert Q_s^{K^-} \rvert p \rangle_h\rvert^2$	Predicted width (*) (Mev)	Experim. width (Mev)
$\left(8, \frac{3}{2}\right) \frac{5}{2}^-$	1765	730	$\frac{2}{15}\rho^2$	33	~ 25
$\left(8, \frac{3}{2}\right) \frac{3}{2}^-$	1670 (a)	800 (a)	$\frac{2}{135}\rho^2$	4	~ 2 (a)
$\left(8, \frac{3}{2}\right) \frac{1}{2}^-$	1750 (b)	1550 (b)	$\frac{16}{27}\rho^2$	~ 300	$\lesssim 50$ (b)
$\left(8, \frac{1}{2}\right) \frac{3}{2}^-$	1670 (a)	850 (a)	$\frac{1}{108}\rho^2$	2,5	
$\left(8, \frac{1}{2}\right) \frac{1}{2}^-$	1750 (b)	1550 (b)	$\frac{1}{54}\rho^2$	~ 10	
$\left(10, \frac{1}{2}\right) \frac{3}{2}^-$	1670 (a)	850 (a)	$\frac{1}{108}\rho^2$	2,5	
$\left(10, \frac{1}{2}\right) \frac{1}{2}^-$	1750 (b)	1550 (a)	$\frac{1}{54}\rho^2$	~ 10	

(*) with $\rho^2 \approx \frac{1}{3}$ from $N^* \longrightarrow N\pi$ decays. a) b) Only one state in Rosenfeld table.

TABLE IV

PREDICTIONS FOR $S = \frac{3}{2}$, Σ DECAYS

$\Sigma_{(\frac{3}{2})}$	MASS (Mev)	DECAY	KINEM. FACTOR (Mev)	$\sum_h \lvert\langle \Sigma \lvert Q_- \rvert \Lambda \rangle_h\rvert^2$	UNCORRECTED WIDTH (Mev)	CORRECTION FACTOR	PREDICTED WIDTH (Mev)	EXPERIM. WIDTH (Mev)
$\frac{5}{2}^-$	1765	$\Lambda(1405)\,\pi$	101	0	0	0,79	0	
		$\Lambda(1520)\,\pi$	36	$\left(\delta^{\beta}\right)^2$	20	0,59	12	~ 18
$\frac{3}{2}^-$	1670	$\Lambda(1405)\,\pi$	64	$\frac{10}{27}\left(\delta^{\beta}\right)^2$	13	0,63	8	2 – 10
		$\Lambda(1520)\,\pi$	14	$\frac{8}{27}\left(\delta^{\beta}\right)^2$	2,3	0,30	0,7	
$\frac{1}{2}^-$	~ 1750	$\Lambda(1405)\,\pi$	280	$\frac{8}{27}\left(\delta^{\beta}\right)^2$	47	0,77	36	
		$\Lambda(1520)\,\pi$	94	$\frac{1}{27}\left(\delta^{\beta}\right)^2$	2	0,52	1	

TABLE V

$\Sigma \in (8)$ $(S)\ J^P$	MASS	KINEM. FACTOR $(GeV)^3$	$\langle \Lambda(1405) \vert Q_5^{\pi^0} \vert \Sigma^0 \ldots \rangle \times \langle \Sigma^0 \ldots \vert Q_5^{K^-} \vert p \rangle$	RESIDUE OF Σ pole (with SIGN) $(GeV)^3$ with $\rho \simeq \frac{1}{\sqrt{3}}$, $\delta_\beta \simeq \frac{3}{4}$	KINEM. FACTOR $(GeV)^3$	$\langle \Lambda(1520) \vert Q_5^{\pi^0} \vert \Sigma^0 \ldots \rangle \times \langle \Sigma^0 \vert Q_5^{K^-} \vert p \rangle$	RESIDUE OF Σ pole (with SIGN) $(GeV)^3$ with $\rho \simeq \frac{1}{\sqrt{3}}$, $\delta_\beta \simeq \frac{3}{4}$
$(\frac{3}{2})\ \frac{5}{2}^-$	1765	27	0	0	19,3	$+\frac{1}{5\sqrt{2}}\,\rho\,\delta^\beta$	+ 1,2
$(\frac{3}{2})\ \frac{3}{2}^-$	1670 [a]	15,5	$-\frac{1}{27}\rho\,\delta^\beta$	− 0,33	9,3	$-\frac{\sqrt{2}}{135}\rho\,\delta^\beta$	$-4,5\times10^{-3}$
$(\frac{1}{2})\ \frac{3}{2}^-$			$\frac{\rho}{36}\left\{\frac{\delta^\beta}{3}+\frac{\delta_0^\alpha}{2}+\frac{\delta_+^\alpha}{2}\right\}\rho$	+		$-\frac{\rho}{36\sqrt{2}}\left\{\frac{\delta^\beta}{6}+\delta_0^\alpha-\frac{\delta_+^\alpha}{2}\right\}\rho$	(−?)
$(\frac{3}{2})\ \frac{1}{2}^-$	∿ 1750 [b]	24	$\frac{4}{27}\rho\,\delta^\beta$	+ 1,3	16,7	$-\frac{\sqrt{2}}{27}\rho\,\delta^\beta$	− 0,38
$(\frac{1}{2})\ \frac{1}{2}^-$			$\frac{\rho}{36}\left\{\frac{\delta^\beta}{6}-\frac{1}{2}\delta_0^\alpha+\delta_+^\alpha\right\}\rho$	(+?)		$\frac{\rho}{18\sqrt{2}}\left\{\frac{\delta^\beta}{3}+\frac{\delta_0^\alpha}{2}+\frac{\delta_+^\alpha}{2}\right\}\rho$	+

TABLE VI

P A R T I E II

-:-:-:-:-:-:-:-:-

SOUS-GROUPES CONNEXES DE POINCARE ET CHAMPS ELECTROMAGNETIQUES

CHAPITRE I - CONNECTED SUBGROUPS OF THE POINCARE GROUP

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ABSTRACT : In view of the determination, up to a conjugation, of connected subgroups of the Poincaré group P , we present in a first paper the following results :

i) Classification of all connected subgroups of dimension one.

ii) Connected subgroups of the Lorentz group L(3,1). The translation group T(4) , the Euclidean group E(3) , the pseudo-Euclidean group E(2,1) .

iii) Maximal connected subgroups of P and their maximal connected subgroups.

An important number of general theorems on inhomogeneous Lie algebras and useful properties are proved.

A physical application concerns the motion of charged particles (tardyons, tachyons or luxyons) in a homogeneous field.

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- C O N T E N T S -

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APPENDIX A

On automorphisms of inhomogeneous Lie algebra.

APPENDIX B

A theorem on semi-simple Lie algebra.

APPENDIX C

A group-theoretical definition of motion of charged particles in a homogeneous field.

I - INTRODUCTION

Looking for all subgroups of the Poincaré group may appear, at first sight, as a problem of little interest for Physics. Nevertheless, many of these subgroups have been investigated in the following sorts of problems.

i) Finding irreducible representations of the Poincaré group in a "given" basis. Such a program consists in choosing a chain of subgroups and diagonalizing invariant operators under these subgroups. For instance, in the so-called canonical representation[1-4], first obtained by Wigner, the translation subgroup plays a particular role since are simultaneously diagonalized the four-momentum and one component of the spin which are invariant operators under translations. Other examples are the representations given in a Lorentz basis[5-8], in an angular momentum basis[9],[10], in an L-S basis[11], in an inh S(2) basis[12],[13]. Related works must be mentioned about the representations of the Lorentz group itself, using different chains of subgroups, namely $O(3,1) \supset O(3) \supset O(2)$ [14], $O(3,1) \supset O(2,1) \supset O(2)$ [15], $O(3,1) \supset E(2)$ [16] and $O(3,1) \supset O(2,1) \supset O(1,1)$ [17-19].

ii) It has been proposed by various authors to use new wave equations for elementary particles, by replacing Minkowski space-time, on which wave functions are defined, by another homogeneous space H of the Poincaré group. Two attempts have been explicitly made, one where H is the Poincaré group itself[20] and one where H is an eight-dimensional space[21-24].

In this kind of models, a given wave equation corresponds to a set of elementary particles (i.e. a <u>reducible</u> representation of the

Poincaré group) leading in a natural way to Regge trajectories as in [22] or to new quantum numbers as in [21],[24]. Some relationship can be found with infinite component wave equations.

It is well-known that homogeneous spaces can be put in a one-to-one correspondence with subgroups. A classification of some homogeneous spaces (namely those which "contain" the Minkowski space and are associated with connected subgroups of the Poincaré group) has been given in [23], by making use of the classification of the Lorentz subgroups given by D. Finkelstein[25](1). Some general features about field theory on homogeneous spaces have been discussed by various people[28],[29].

iii) Representations of subgroups of the Poincaré group have been used when physical systems are placed in an external field with a given symmetry such investigations have been made in the case of a constant and uniform electromagnetic field[30,31] and the field of a circularly polarized plane wave[32,33]. It would be interesting to have a complete classification of all electromagnetic fields carrying a symmetry. For such a purpose, a perfect knowledge of the subgroups lattice of the Poincaré group would be desirable. Obviously, a similar discussion can be made with the Galilei group replacing the Poincaré group[34]. This kind of applications seem very promising for specific problems as it has been shown by Richard[35]

(1) Since the work of Finkelstein, the classification of the Lorentz subgroups has been given independently by Winternitz and Fris[26]. We give in Section IV a rigorous derivation of the Lorentz subgroups lattice which agrees with Winternitz results. After completion of the present work, we learned that a similar classification has been carried out recently by Shaw[27].

iv) the Compton effect in a laser beam.

iv) We must also mention, as a problem involving a peculiar subgroup of the Poincaré group (namely the inhomogeneous S(2) group) that of infinite momentum frames. We refer the reader to paper[13] where some of irreducible representations of this group have been given. The reduction of a non zero-mass representation with respect to this subgroup is found to be equivalent to a formal contraction process from the usual Wigner canonical representation[12].

The present work can be considered as a step towards a complete knowledge of subgroups of the Poincaré group. On this subject, we must refer the reader to the work by Jocs and Schrader[36] and that of Janner et al [37,38].

Classifying all connected subgroups of a given connected Lie group G is equivalent to classifying all subalgebras $\mathcal{g}$ of G. This property is a consequence of the following theorem :

Theorem

i) If H is a Lie subgroup of G, its Lie algebra $\mathcal{h}$ is a subalgebra of the Lie algebra $\mathcal{g}$ of G.

ii) Each subalgebra of $\mathcal{g}$ is the Lie algebra of exactly one connected Lie subgroup of G.

(For the proof, see, for instance, ref. [40], p. 102).

We are interested in subgroups up to a conjugation. It is well known that the conjugation is an equivalence relation, two subalgebras being equivalent if there exists a transformation of G which maps one onto the other. However, while all subalgebras of $\mathcal{g}$ form a lattice (we shall denote this lattice $\mathrm{Latt}(\mathcal{g})$), the conjugacy classes do not constitute in general a lattice, but only a partially ordered set (p.o.s) which will be denoted $\Pi(\mathcal{g})$.

In order to obtain $\mathcal{T}(\mathcal{P})$, $\mathcal{P}$ being the Poincaré Lie algebra, we can think of various systematic ways. Each of them has its own advantages but we were not able to find one which was not tedious. Fortunately, it is possible, with the aid of simple theorems, to get immediate information on parts of $\mathcal{T}(\mathcal{P})$. This information concerns

- the one dimensional subalgebras of $\mathcal{P}$, that is the bottom of $\mathcal{T}(\mathcal{P})$ (section II). The results are summarized in table II-5, the necessary notations are given in section I and section II.

- the maximal subalgebras of $\mathcal{P}$, that is the top of $\mathcal{T}(\mathcal{P})$ see table VI-I.

- the part of $\mathcal{T}(\mathcal{P})$ which only contains translations (see table IV-I).

- $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}(E(3))$ (where $\mathcal{L}$ and $E(3)$ are respectively the Lorentz and Euclidean algebra) which are shown to be subsets of $\mathcal{T}(\mathcal{P})$ in the sense explained above (see tables IV-2 and V-I).

- $\mathcal{T}(E(2,1))$ (where $E(2,1)$ is the Poincaré group of a three dimensional space-time) which is <u>almost</u> contained in $\mathcal{T}(\mathcal{P})$ (see table V-2).

Here is a résumé of our approach :

i) In section II, all subalgebras of dimension one are obtained by considering the action of the Poincaré group on the ray space associated with the Lie algebra.

ii) In order to build the subalgebras of larger dimension, we need a complete knowledge of $\mathcal{T}(\mathcal{L})$ and the action of each Lorentz connected subgroup on space-time. This is done in section IV. This provides us with $\mathcal{T}(\mathcal{L})$ and $\mathcal{T}(\mathcal{T}_4)$ (corresponding to the Lorentz and the translational algebras respectively) which are subsets of $\mathcal{T}(\mathcal{P})$.

iii) The subsets corresponding to $E(3)$ and $E(2,1)$ are derived in section V.

iv) Section VI is devoted to the search of "sons" and "grandsons" of the Poincaré group, that is the maximal connected subgroups of P and

their maximal connected subgroups, respectively.

v) All general theorems are put together in Section III. Other ones of more general interest are given in Appendices A and B.

vi) Finally, we have given an amusing physical interpretation of the one-dimensional connected subgroups in Appendix C.

The construction $\mathcal{T}(\mathcal{P})$ will be completed in a forthcoming paper.

We complete this introduction by recalling some properties of the Poincaré group and by precising our terminology and notation.

a) Each element of the Poincaré group (a,L) can be written in a unique way as a Lorentz transformation L followed by a translation a, with the following group law :

$$(a',L')(a,L) = (a' + L'a, L'L)$$

where

$$(L'a)^\mu = L'^\mu{}_\nu \, a^\nu \,. \tag{I.1}$$

b) As it is well-known, the study of the Lorentz group is conveniently replaced by that of its covering group $SL(2,C)$. We define the Pauli 2×2 - matrix :

$$\sigma(x) = x^0 - \vec{\sigma}.\vec{x} \tag{I.2}$$

associated with a real four-vector x. To each transformation L corresponds a matrix $\Lambda \in SL(2,C)$, defined up to a sign, such that :

$$\Lambda \, \sigma(x) \, \Lambda^+ = \sigma(Lx) \,. \tag{I.3}$$

The couple $[\sigma(a),\Lambda]$ denotes the generic element of the covering group $\bar{P}$ whose group law is :

$$[\sigma(a'),\Lambda'][\sigma(a),\Lambda] = [\sigma(a' + L'a), \Lambda'\Lambda] \,. \tag{I.4}$$

c) The Lorentz group is known to be isomorphic to $SO(3,C)$. Let $\vec{F} = \vec{B} + i\vec{E}$ be a complex three vector. The 2x2-matrix $\vec{\sigma}.\vec{F}$ is traceless and its determinant is $\vec{F}^2$. Any element $\Lambda \in SL(2,C)$ acting linearly as follows :

$$\vec{\sigma}.\vec{F} \rightarrow A(\vec{\sigma}.\vec{F})A^{-1}$$

leaves invariant the trace and the determinant. It is easily shown that $SL(2,C)$ is a two-valued representation of $SO(3,C)$ which proves its isomorphism with the Lorentz group. More precisely, if $R \in SO(3,C)$

$$A(\vec{\sigma}.\vec{F})A^{-1} = \vec{\sigma}.(R\vec{F}) \quad (I.5)$$

This formula expresses the action of the Lorentz group on its Lie algebra through the convention :

$$\vec{E} = (F^{01}, F^{02}, F^{03})$$

$$\vec{B} = (F^{23}, F^{31}, F^{12}) \quad (I.6)$$

which establishes the connection between the vector $\vec{B} + i\vec{E}$ and the element $\frac{1}{2} F^{\mu\nu} M_{\mu\nu}$ of the Lie algebra, where the $M_{\mu\nu}$'s are the usual generators of the Lorentz group (each tensor F of components $F^{\mu\nu} = -F^{\nu\mu}$ corresponding to a single element of the Lie algebra).

d) Action of the Poincaré group on its Lie algebra :

We define the following four-dimensional two-valued representation :

$$(a,L) \rightarrow \pm \begin{bmatrix} \Lambda & -\frac{i}{2}\sigma(a)\,\Lambda^{+-1} \\ 0 & \Lambda^{+-1} \end{bmatrix} . \quad (I.7)$$

Let us consider an infinitesimal transformation ; $(a,L) \simeq 1 - i(x,F)$

In this representation, the element (x,F) of the Lie algebra reads :

$$(x,F) \rightarrow \frac{1}{2} \begin{bmatrix} \vec{\sigma}.\vec{F} & \sigma(x) \\ 0 & (\vec{\sigma}.\vec{F})^{+} \end{bmatrix} . \quad (I.8)$$

The action of P (or $\vec{P}$) on its Lie algebra readily follows

$$(a,L)(x,F)(a,L)^{-1} = (Lx + a.F', F') \quad (I.9)$$

where $\vec{F'} = R\vec{F}$ (R is the element of $SO(3,C)$ associated with L according to Eq (I.5), and $(a.F')^{\mu} = a_{\nu} F'^{\nu\mu}$).

It is important to note that the action on the Lie algebra adjoint representation is quite different of the one on its dual vector space (coadjoint representation). As an example, let us consider how the element : $(x,F) = x^{\mu}P_{\mu} + \frac{1}{2} F^{\mu\nu} M_{\mu\nu}$ is transformed under a translation a :

$$\begin{cases} x \rightarrow x + a.F \\ F \rightarrow F \end{cases} \quad (I.10)$$

$$(x,F) \rightarrow (x^{\mu} + a_{\nu} F^{\nu\mu}) P_{\mu} + \frac{1}{2} F^{\mu\nu} M_{\mu\nu}$$

$$= x^{\mu}P_{\mu} + \frac{1}{2} F^{\mu\nu}(M_{\mu\nu} + a_{\mu}P_{\nu} - a_{\nu}P_{\mu})$$

from which we deduce the action on the components of an element of the dual space :

$$\begin{cases} P_{\mu} \rightarrow P_{\mu} \\ M_{\mu\nu} \rightarrow M_{\mu\nu} + a_{\mu}P_{\nu} - a_{\nu}P_{\mu} . \end{cases} \quad (I.11)$$

In the following, we will make use of both adjoint and coadjoint representations.

II - CONNECTED SUBGROUPS OF DIMENSION ONE

LEMMA[2]

Two elements of the Lorentz group are conjugate under a Poincaré transformation if and only if they are conjugate under a Lorentz transformation.

The proof is a direct consequence of the multiplication law (I,1). Let $(0,L_1)$ and $(0,L_2)$ be two conjugate elements of the Lorentz group. One gets

$$(II.1) \qquad (a,L)(0,L_1)(a,L)^{-1} = (0,L_2) \Leftrightarrow LL_1L^{-1} = L_2$$

Corollary[2]: The p.o.s. $\mathcal{H}(L)$ of Lorentz subgroups is a subset of the p.o.s. $\mathcal{H}(\mathcal{P})$ of Poincaré subgroups. Obviously, this result is also true for the p.o.s. $\mathcal{H}(\mathcal{L})$ ($\mathcal{L}$ = Lie algebra of L) of the corresponding Lie algebras. This property allows us to investigate $\mathcal{H}(\mathcal{L})$ separately (Section IV). In the present section, we intend to bring our attention on the one-dimensional Lorentz subalgebras.

Any non-zero element s of the Lie algebra $\mathcal{G}$ of a Lie group G, generates a one-dimensional connected subgroup. Two non-zero elements s and s' of $\mathcal{G}$ generate the same subgroup if and only if they are collinear, i.e. belong to the same ray. Therefore, we must look for the action of the group G on the projective space $\hat{\mathcal{G}}$ of $\mathcal{G}$. Finding the one-dimensional subalgebras of $\mathcal{G}$, up to a conjugation, is equivalent to finding the orbits of G on $\hat{\mathcal{G}}$.

(2) It is clear that the lemma and its corollary are valid in more general cases, in particular for all inhomogeneous groups (see Section III).

1- ONE DIMENSIONAL SUBALGEBRAS OF THE LORENTZ ALGEBRA $\mathcal{L}$ AND THEIR STABILITY GROUPS

According to the preceding analysis, we must look for orbits of the Lorentz group L on the projective space $\hat{\mathcal{L}}$ of skew-symmetric tensors F of order two. Let us define the two invariants :

$$(II.2) \qquad \begin{aligned} \Delta(F) &= \tfrac{1}{2} F_{\mu\nu}F^{\mu\nu} = \vec{B}^2 - \vec{E}^2 . \\ \Phi(F) &= \tfrac{1}{4} \epsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda} = 2\vec{E}\cdot\vec{B} \end{aligned}$$

related with the length of $\vec{F}$ by

$$(II.3) \qquad \vec{F}^2 = \Delta(F) + i\,\Phi(F) .$$

The action of $SL(2,C)$ on its Lie algebra is given by

$$(II.4) \qquad \vec{\sigma}.\vec{F}' = A\,(\vec{\sigma}.\vec{F})\,A^{-1} .$$

By a suitable $SL(2,C)$ transformation, it is always possible to map $\vec{\sigma}.\vec{F}$ on a diagonal matrix or on a matrix of Jordan form $\begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ [36]. Due to the fact that both trace and determinant are conserved under such a transformation, the final matrices read : $\begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ respectively, with: $-\lambda^2 = \Delta + i\,\Phi$. Therefore[3], the orbits on $\hat{\mathcal{L}}$ are in one-to-one correspondence with points in the plane (Δ,Φ) (except, of course, the null vector of $\mathcal{L}$ which constitute a single orbit). In fact, we are interested in orbits on $\hat{\mathcal{L}}$, the ray space associated with $\mathcal{L}$. It is clear that if $\vec{F}$ and $\vec{F}'$ belong to the same ray, their squared lengths have a real ratio, or are both zero. Consequently, the orbits on $\hat{\mathcal{L}}$ are described in the (Δ,Φ) plane in the following way :

i) one orbit is characterized by $\Delta = \Phi = 0$.

ii) each other orbit is represented by a half-straightline δ_f :

(3) It readily follows that $SL(2,C)$ has the complex spheres $\vec{F}^2 = \text{constant}$ as orbits in $\mathbb{C}^3 - \{0\}$.

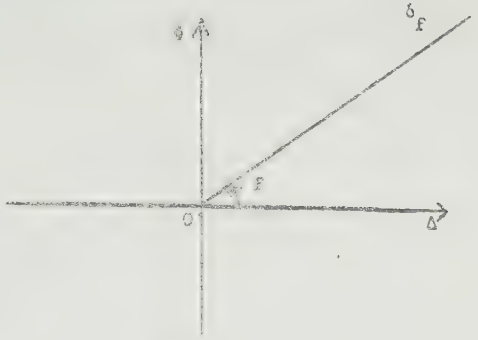

Fig. II.1

labelled by the angle f where[4] $0 \leq f < 2\pi$ and

$$\text{(II.5)} \qquad \text{tang } f = \frac{2\,\vec{E}.\vec{B}}{\vec{B}^2 - \vec{E}^2}$$

If we denote by $M_{\mu\nu}$ the generators of the Lorentz group and define

$$\text{(II.6)} \qquad \begin{aligned} \vec{J} &= (M^{23}, M^{31}, M^{12}) \rightarrow \tfrac{1}{2}\vec{\sigma} \\ \vec{K} &= (M^{01}, M^{02}, M^{03}) \rightarrow -\tfrac{i}{2}\vec{\sigma} \end{aligned}$$

the most general element of $\mathcal{L}$ may be written in the form

$$\text{(II.7)} \qquad \tfrac{1}{2}\omega^{\mu\nu} M_{\mu\nu} = \vec{B}.\vec{J} - \vec{E}.\vec{K}$$

Typical examples of generators of non-conjugate Lie algebras are given by

$$\text{(II.8)} \quad \begin{cases} \text{i)} & A_y = K_y + J_x \quad \text{(point 0 in Fig. II.1)} \\ \text{ii)} & J_f = \cos\frac{f}{2} J_z - \sin\frac{f}{2} K_z \quad \text{(half straightline } \delta_f \text{ in Fig. II.1)} \end{cases}$$

It will appear later that the values $f = 0$ and π play a particular role. Therefore, we will distinguish the four following boxes of one-dimensional connected subgroups, namely :

(4) Note the slight mistake in ref [24] where f is supposed to obey $0 \leq f < \pi$.

SO(2) | L(1,1)[5] | A(1) | F_f

generated by J_z, K_z, A_y and J_f respectively where the angle f is such that $\sin f \neq 0$. The three first boxes correspond to the cases where $\Phi(F) = 0$, (i.e. where $\det(F_{\mu\nu}) = [\Phi(F)]^2 = 0$) and $\Delta(F)$ positive, negative and zero, respectively. Each of them represents a unique group up to a conjugation. We have drawn a different kind of box for the Finkelstein groups because to each value of f corresponds a group which is unique, up to a conjugation.

Let us now look for the stability groups of these one parameter subalgebras of $\mathcal{L}$ which will be useful in many parts of this work. Let $\Lambda = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ with $\alpha\delta - \beta\gamma = 1$ be the general matrix of $SL(2,C)$. The action of Λ on $\mathcal{L}$ can be written :

$$\text{(II.9)} \qquad \Lambda\,\vec{\sigma}.\vec{F}\,\Lambda^{-1} = \vec{\sigma}.(R\vec{F}) \quad \text{with } R \in SO(3,C).$$

The stability group of a one parameter subalgebra is the set of all matrices Λ such that :

$$\text{(II.10)} \qquad \Lambda\,\vec{\sigma}.\vec{F}\,\Lambda^{-1} = \lambda\,\vec{\sigma}.\vec{F} \quad \text{with } \lambda \in \mathbb{R} - \{0\}.$$

According to (II.8), we have only two cases to consider :

i) $\vec{F} = (1,-i,0)$: the stability group is the set of all matrices $\begin{bmatrix} \alpha & 0 \\ \gamma & \alpha^{-1} \end{bmatrix}$ with : $\alpha^{-2} = \lambda$, $\lambda \in \mathbb{R} - \{0\}$, $\gamma \in C$.

The Lie algebra of this group is generated by K_z, $A_x = K_x - J_y$, and $A_y = K_y + J_x$. This group will be denoted $Th(2)$[6], the Thales group

(5) $L(p,1)$ denotes the connected component of the Lorentz group associated with a $(p+1)$ - dimensional space time. The group F_f will be called Finkelstein groups. Therafter, the same symbol will be used to denote the Lie group and its Lie algebra.

(6) A representation of $Th(2)$ is given by the group of 3x3 real matrices of the form $\begin{vmatrix} \pm e^{\alpha} & 0 & a_1 \\ 0 & \pm e^{\alpha} & a_2 \\ 0 & 0 & 1 \end{vmatrix}$

(T for translations, h for homothethy) since it is isomorphic to the group generated by translations and dilatations in the 2-dimensional real-vector space. This group is two-sheeted, the connected part corresponding to positive dilatations.

ii) $P = (0,0,F_z)$: the stability group is composed of two sheets : the first one contains the matrices $\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix}$ corresponding to $\lambda = -1$, and the second one the matrices $\begin{bmatrix} 0 & \beta \\ -\beta^{-1} & 0 \end{bmatrix}$ corresponding to $\lambda = 1$, where $\alpha, \beta \in \mathbb{C}$. This group is generated by J_z and K_z. Its connected component is isomorphic to $SO(2) \times L(1,1)$ [7].

Let us now sum up all the results of this section in Table II.1

One-dimensional subalgebras in $\mathcal{L}$.

generator conjugate to	corresponding invariants	symbol	stability group [7]
$J_f = \cos\frac{f}{2} J_z - \sin\frac{f}{2} K_z$	f $(\sin f \neq 0)$	F_f	$[SO(2) \times L(1,1)] \square Z_2$ generated by J_z and K_z
J_z	$f = 0$	$SO(2)$	
K_z	$f = \pi$	$L(1,1)$	
$A_y = K_y + J_x$	$\Delta = \mathfrak{s} = 0$ $(P \neq 0)$	$A(1)$	$Th(2)$ generated by K_z , A_x , A_y (two-sheeted)

TABLE II.1

(7) The nonconnectedness corresponds to simultaneous parity transformations in (x,y) and (t,z) planes. This transformation is contained in the connected Lorentz group as the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ describing a rotation of angle π around the z-axis.

2 - ONE DIMENSIONAL POINCARE SUBALGEBRAS AND THEIR CHARACTERIZATION BY INVARIANTS

In order to find the one-dimensional Poincaré subalgebras, we will use a similar way, that is first construct invariants, second try to define some "canonical" element on each orbit on $\hat{\mathcal{P}}$.

Invariants on $\hat{\mathcal{P}}$ under $\underline{P}$

We shall first consider invariants on $\mathcal{P}$ under $\underline{P}$; it will be then easy to deduce those on $\hat{\mathcal{P}}$ under $\underline{P}$.

From an element (x,F) , we can build four [8] "continuous" independent Lorentz scalars, namely, $\Delta(F)$, $\mathfrak{s}(F)$, $(F^*.x)^2 = U$, x^2 , and "discrete" invariants such that $\operatorname{sgn} x^0$, $\operatorname{sgn}(F^*.x)^0$, ...

Let us consider all real functions h of the four variables Δ , $\mathfrak{s}$, U , x^2 which are invariants under translations. Under an infinitesimal translation a , the function h becomes :

$$(II.13) \quad h(\Delta,\mathfrak{s},U,x^2) \rightarrow h(\Delta,\mathfrak{s},U,x^2) + \mathfrak{s}\frac{\partial h}{\partial U} a.F^*.x + 2\frac{\partial h}{\partial x^2} a.F.x - \frac{\mathfrak{s}^2}{4}\frac{\partial h}{\partial U} a^2 + \frac{\partial h}{\partial (x^2)} (F.a)^2 + \frac{\mathfrak{s}^2}{2}\frac{\partial^2 h}{\partial U^2} (a.F^*.x)^2 + O(a^3) .$$

(8) It is well-known that the Lorentz group has only two invariants $\mathfrak{s}$ and Δ . Moreover, we have the following identities :

$$(II.11) \quad (x.F^*) \cdot (F.x) = -\tfrac{1}{2}\mathfrak{s}(F)\, x^2$$

and

$$(II.12) \quad x^2 \Delta(F) = (F.x)^2 - (F^*.x)^2$$

where $(F.x)^\mu = F^{\mu\nu} x_\nu$.

These identities restrict the number of "continuous" invariants.

The invariance condition on h for all translation a implies [9] :

$$(II.14) \qquad \vartheta \frac{\partial h}{\partial U} F^*.x + 2 \frac{\partial h}{\partial (x^2)} F.x = 0$$

$$(II.15) \qquad -\frac{\vartheta^2}{4} \frac{\partial h}{\partial U} a^2 + \frac{\partial h}{\partial (x^2)} (F.a)^2 + \frac{\vartheta^2}{2} \frac{\partial^2 h}{\partial U^2} (a.F^*.x)^2 = 0 \quad , \quad \forall a .$$

Replacing h by Δ and ϑ, successively, in Eqs. (II.14) and (II.15) proves that Δ and ϑ are both invariant, a result which could be readily obtained from the lemma given at the beginning of the present section.

Case 1 : $\vartheta(F) \neq 0$

F is nonsingular. Therefore, provided $x \neq 0$, we get: $F.x \neq 0$ and $F^*.x \neq 0$. By taking the scalar product of the l.h.s. of Eq. (II.14) by $F.x$ and $F^*.x$, successively, it is not difficult to prove, with the aid of the identities (II.12) and (II.13) that ϑ and Δ are the only continuous invariants.

Case 2 : $\vartheta(F) = 0$, $F \neq 0$

There exists a vector a such that $(F.a)^2 \neq 0$. Therefore according to Eq. (II.15), h cannot depend on x^2. The only continuous invariants are Δ and U.

Case 3 : $F = 0$ (pure translation)

The only continuous invariant is obviously x^2.

Now the invariants on $\hat{\mathcal{P}}$ are those functions $\hat{h}$ which satisfy also the condition

$$(II.16) \qquad \hat{h}(x,F) = \hat{h}(\lambda x, \lambda F)$$

where $\lambda \neq 0$. In other words, $\hat{h}$ must be a homogeneous function of degree

(9) We look for the terms of first and second order only. However, it is easy to see that the obtained functions are invariant at any order.

zero in x and F. It is clear that if ϑ and Δ are not both zero, the ration ϑ/Δ is an invariant.

Case 1 : $\vartheta(F) \neq 0$

Since Δ and ϑ are the only continuous invariants on $\hat{\mathcal{P}}$, $\tan f = \vartheta/\Delta$ is the only continuous invariant on $\hat{\mathcal{P}}$. Note that all values of f are admitted except those for which ϑ vanishes, that is $f = 0$ or π ; the fact that there exists no discrete invariant will be proved later.

Case 2 : $\vartheta = 0$

The only continuous invariant is U/Δ^2.

We already noted that we have three subcases, namely $f = 0$, $f = \pi$ and $\Delta = 0$, corresponding to the kind of tensor F. Besides these three "discrete" invariants, one easily verifies that the kind of the vector $F^*.x$ is also an invariant. Therefore, we are led to a rectangular table where some boxes have been suppressed because some incompatibilities between respective kinds of F and $F^*.x$.

Kind of F \ Kind of $F^*.x$	time like (future)	time like (past)	light like (future)	light like (past)	space like	zero
$\Delta > 0$ ($f = 0$)	$\frac{U}{\Delta^2}$	$\frac{U}{\Delta^2}$	1 orbit	1 orbit	$\frac{U}{\Delta^2}$	1 orbit
$\Delta < 0$ ($f = \pi$)					$\frac{U}{\Delta^2}$	1 orbit
$\Delta = 0$			1 orbit	1 orbit	1 orbit	1 orbit

TABLE II.2

Case 3 : $F = 0$ (pure translation)

It is quite clear that there is no continuous invariant and that the only discrete invariants correspond to the three kinds of four-vectors x, namely time-like ($x^2 > 0$), light-like ($x^2 = 0$) and space-like ($x^2 < 0$). These three cases correspond to the one-dimensional subgroups $T^+(1)$, $T^o(1)$ and $T^-(1)$ respectively.

It is not yet proved that our classification of orbits (versus invariants) is complete. In fact, we are not sure that all discrete invariants have been obtained. Nevertheless, the following direct investigation will furnish a proof of the above results.

Orbits on $\hat{\mathcal{P}}$ under $\underline{P}$

The method which we use is similar to that we adopted in the case of $\hat{\mathcal{L}}$, that is we want to define on each orbit a "canonical" element, that is some element of simple form. We will proceed as follows. Let $\widehat{(x,F)}$ be an element of $\hat{\mathcal{P}}$. By using a Lorentz transformation, one can first put F into its canonical form F' (diagonal or Jordan Form)[10]. This new element can be transformed in the "canonical" element with the aid of a Poincaré transformation, the Lorentz part of which belonging to the stabilizer of F'. The group of elements of $\underline{P}$ which leave invariant the Lorentz part F' (up to a multiplicative coefficient λ) of $\widehat{(x',F')}$ are of the form (a,L_{p_o}) with L_{p_o} belonging to the stability group of $\hat{F}'$ (see Table II.1) and a being a translation. We have the formula :

$$(a,L_p)\widehat{(x,F)}(a,L_p)^{-1} = \widehat{(L_p x + \lambda a F\ ,\ \lambda F)} \qquad \lambda \in R - \{0\} . \tag{II.17}$$

(10) Note that the case $F = 0$ cannot be longer discarded when x is not zero.

We note in particular that (x,F) is transformed into $(x + aF,F)$ by action of $(a,\mathbb{1})$.

Let us then consider the different cases following the canonical forms of F :

Case 1 : $\mathcal{F} \neq 0$ (Lorentz generator J_f)

The components of $a.F$ are : $a^3 \sin\frac{f}{2}$, $a^2\cos\frac{f}{2}$, $-a^1\cos\frac{f}{2}$, $a^o\sin\frac{f}{2}$ which show that for each (x,F) with F corresponding to an element J_f of $\mathcal{L}$, there exists a transformation $(a,\mathbb{1})$ such that :

$$(a,1)(x,F)(-a,\mathbb{1}) = (0,F) . \tag{II.18}$$

Consequently, we have the only class F_f.

Case 2 : $\mathcal{F} = 0$

i) $f \neq 0$ (Lorentz generator J_z) :

With the aid of a translation, we can drop the components x^1 and x^2 of vector x. A boost along the z-axis, (followed, if necessary, by a rotation of angle π around the y-axis) permits us to map the vector x on a given vector of a given type. We are led to the five sorts of classes of one-dimensional Lie subalgebras which are given in the first row of Table II.3 (compare this row to the corresponding row of Table II.1)

ii) $f = \pi$ (Lorentz generator K_z) :

The use of a translation followed by a rotation around the z-axis leads us to the two sorts of classes which are given in the second row of Table II.3 .

iii) $\Lambda \neq 0$ (Lorentz generator Λ_y) :

After a translation, we perform a transformation generated by Λ_y, a boost along the z-axis and, if necessary, a rotation of angle π around the same axis. We are led to the four types of groups which are given in the third row of Table II.3

Kind of F \ Kind of x	time-like (future)	time-like (past)	light-like (future)	light-like (past)	space-like	zero
$f = 0$	$H^{\pm}_{\omega}$ $\omega < 0$ | $\omega > 0$ $(\omega J_z + P_z)$		$H^{\uparrow}$ $(J_z+P_t+P_z)$	$H^{\downarrow}$ $(J_z+P_t-P_z)$	H^{-}_{ω} $\omega > 0$ (ωJ_z+P_t)	$SO(2)$ (J_z)
$f = \pi$					V_{σ} $\sigma > 0$ (σK_z+P_x)	$L(1,1)$ (K_z)
$\Delta = 0$			$A^{\uparrow}(1)$ $(A_y + P_x)$	$A^{\downarrow}(1)$ $(A_y - P_x)$	$A^{-}(1)$ $(A_y+P_t+P_z)$	$A(1)$ (A_y)

TABLE II.3

In our notations H is put for helicity, V for velocity and the upscripts $+$, $-$, $\uparrow$, $\downarrow$ correspond to the kind of the vector $F^*.x$.

Case 3 : $f = 0$

This case has already been completly considered. We can put the results in the following table :

Kind of x \ Kind of F	Time-like	Light-like	Space-like
$F = 0$	$T^{+}(1)$ (P_t)	$T^{o}(1)$ $(P_t - P_z)$	$T^{-}(1)$ (P_z)

TABLE II.4

The upscripts characterize the kind of x.

We can now look for the stability groups of these one-parameter subalgebras of $\mathcal{P}$. According to Eq. (I.9), the stability group of a one-parameter subalgebra characterized by the element $\widehat{(x,F)}$ of $\hat{\mathcal{P}}$, is the set of elements (a,L) such that :

$$\begin{aligned} \sigma(Lx + aF^*) &= \lambda\, \sigma(x) \\ \vec{\sigma}.\vec{F}' &= \lambda\, \vec{\sigma}.\vec{F} \end{aligned} \tag{II.19}$$

with $\lambda \in R - \{0\}$.

It follows that if (a,L) belongs to the stability group of $\widehat{(x,F)}$, L belongs to the stability group L_F of $\hat{F}$ in $\underline{L}$. Therefore we have to consider the equation

$$\sigma(L_F x + \lambda a F) = \lambda\, \sigma(x) \tag{II.20}$$

where L_F belongs to $\underline{L}_F$.

Following the classification of the stability groups of the one parameter subalgebras of $\mathcal{L}$ given in Table II.1, we can decompose our study into three parts corresponding to the three forms of F ($\vec{F} = (1,-i,0)$; $\vec{F} = (0,0,F_z)$; $\vec{F} = 0$). In each case we have to consider the different types of x as they appear in Tables II.3 and II.4. Our results are summarized in Table II.5

It is amusing that all these subgroups have a physical interpretation (See Appendix C).

ONE DIMENSIONAL SUBALGEBRAS IN $\mathcal{P}$

Representative element	Corresponding invariants	Symbol	Stability group and its generators
J_f	f : $\sin f \neq 0$	F_f	$[SO(2)\times L(1,1)] \square Z_2$: $\{J_z, K_z\}$
J_z	$F^*.x = 0$	$SO(2)$	$[SO(2)\times E(1,1)] \square Z_2$: $\{J_z, K_z, P_t, P_z\}$
$\omega J_z + P_t$; $\omega > 0$	$\left(\frac{F^*.x}{\Delta}\right)^2 = -\omega^{-2}$	H^-_ω	$T^+(2)\times SO(2)$: $\{J_z, P_t, P_z\}$
$\omega J_z + P_z$; $\omega \neq 0$	$\left(\frac{F^*.x}{\Delta}\right)^2 = -\omega^{-2}$, $\mathrm{sgn}(F^*x)^0 = -\mathrm{sgn}\omega$	H^+_ω	$[T^+(2)\times SO(2)] \square Z_2$: $\{J_z, P_t, P_z\}$
$J_z - (P_t - P_z)$	$\mathrm{sgn}(F^*.x)^0 = +$, $\left(\frac{F^*.x}{\Delta}\right)^2 \neq 0$	$H^\uparrow$	$T^+(2) \times SO(2)$: $\{P_z, P_t, P_z\}$
$J_z + (P_t - P_z)$	$\mathrm{sgn}(F^*.x)^0 = -$, $(F^*x) \neq 0$	$H^\downarrow$	
K_z	$\left(\frac{F^*.x}{\Delta}\right)^2 = 0$	$L(1,1)$	$[E(2)\times L(1,1)] \square Z_2$: $\{K_z, J_z, P_x, P_y\}$
$\sigma K_z + P_y$; $\sigma^2 > 0$	$\left(\frac{F^*.x}{\Delta}\right)^2 = -\sigma^{-2}$	V_σ	$T^-(2)\times L(1,1)$: $\{K_z, P_x, P_y\}$
A_y	$F^*.x = 0$	$A(1)$	$T^0(2) \square Th(2)$: $\{K_z, A_x, A_y, P_x, P_t - P_z\}$
$A_y + P_t + P_z$	$(F^*.x)^2 = -1$	$A^-(1)$	$T^0(2) \times A^-(1)$: $\{A_y + P_t + P_z, P_x, P_t - P_z\}$
$A_y + P_x$	$\mathrm{sgn}(F^*.x)^0 = +$, $(F^*.x)^2 \neq 0$	$A^\uparrow(1)$	$[T^0(2) \square A^\uparrow(2)] \square Z_2$: $\{A_x + P_y, A_y + P_x, P_x, P_t - P_z\}$
$A_y - P_x$	$\mathrm{sgn}(F^*.x)^0 = -$, $(F^*.x) \neq 0$	$A^\downarrow(1)$	$[T^0(2) \square A^\downarrow(2)] \square Z_2$: $\{A_x - P_y, A_y - P_x, P_x, P_t - P_z\}$
P_t	$\mathrm{sgn}\, x^2 = +$	$T^+(1)$	$T^+(1) \times E(3)$: $\{P_t, \vec{P}, \vec{J}\}$
$P_t + P_z$	$\mathrm{sgn}\, x^2 = 0$	$T^0(1)$	$InS(2)$: $\{P_t, \vec{P}, J_z, K_z, A_x, A_y\}$
P_z	$\mathrm{sgn}\, x^2 = -$	$T^-(1)$	$T^-(1) \times E(2,1)$: $\{P_z, P_t, P_x, P_y, J_z, K_x, K_y\}$

TABLE II.5

N.B. In our notation of the stability groups, the discrete group $Z_2 = (1,-1)$ corresponds to a rotation of π around the y-axis.

III - CONNECTED SUBGROUPS OF INHOMOGENEOUS LIE GROUPS

A - GENERAL PROPERTIES

Hereafter, we will call any semi-direct product $\underline{P} = \underline{T} \square \underline{L}$, where $\underline{T}$ is an Abelian Lie group, an inhomogeneous Lie group. The elements of $\underline{P}$ will be denoted (a,L) with the group law :

(III.1) $$(a',L')(a,L) = (a'+L'.a, L'L) .$$

It follows[11] that the elements $(0,L)$ form a subgroup $\underline{L}$ of $\underline{P}$ and the elements $(a,1)$ form a subgroup $\underline{T}$ of $\underline{P}$.

In the same way, the Lie algebra $\mathcal{P}$ of $\underline{P}$ will be denoted by $\mathcal{P} = \tau \square \mathcal{L}$. Under an inner automorphism of $\underline{P}$, τ is unchanged and $\mathcal{L}$ becomes $\mathcal{L}_1$. It follows that $\mathcal{P}$ can now be written in the form $\tau \square \mathcal{L}_1$. Any inner automorphism of $\underline{P}$ can be considered as a product of a homogeneous transformation (element of $\underline{L}$) by a translation (element of $\underline{T}$); all forms $\mathcal{P} = \tau \square \mathcal{L}_1$ are obtained through a translation automorphism. Obviously, as a vector space, $\mathcal{P}$ is the direct sum $\tau + \mathcal{L}$ of the vector spaces τ and $\mathcal{L}$. The inhomogeneous structure is expressed by the relations :

(III.2) $$\begin{cases} [\mathcal{L},\mathcal{L}] \subseteq \mathcal{L} \\ [\mathcal{L},\tau] \subseteq \tau \\ [\tau,\tau] = 0 . \end{cases}$$

(11) Note that $\underline{L}$ cannot be confused with the factor group $\underline{P}/\underline{T}$ and similarly $\mathcal{L}$ with $\mathcal{P}/\tau$.

(We will make use of both signs $\subseteq$ and $\subset$ in order to distinguish between inclusion in the large and in the strict sense).

Let $\mathcal{A}$ be a proper subalgebra of $\mathcal{P}$, i.e.

(III.3) $$\mathcal{A} \subset \mathcal{P} \qquad \mathcal{A} \neq 0$$
$$[\mathcal{A},\mathcal{A}] \subseteq \mathcal{A}$$

Let us define :

((III.4) $$\mathcal{L}_A = \mathcal{A} \cap \mathcal{L}$$
$$\mathcal{T}_A = \mathcal{A} \cap \mathcal{T}$$

The following relations readily follow from (III.2, 3 , 4)

(III.5) $$\begin{cases} [\mathcal{L}_A,\mathcal{L}_A] \subseteq \mathcal{L}_A \\ [\mathcal{L}_A,\mathcal{T}_A] \subseteq \mathcal{T}_A \\ [\mathcal{T}_A,\mathcal{T}_A] = 0 \end{cases}$$

which prove that $\mathcal{T}_A + \mathcal{L}_A$ is a Lie subalgebra of $\mathcal{A}$

(III.6) $$\mathcal{T}_A + \mathcal{L}_A \subseteq \mathcal{A} \subset \mathcal{P}$$

We will adopt the following terminology :

Any subalgebra of the form $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A$ and its conjugates are said to be <u>symmorphic</u>. It can be easily proved that any symmorphic subalgebra can be put in a <u>manifestly symmorphic form</u> (i.e., in the form $\mathcal{T}_A + \mathcal{L}_A$) through a translation automorphism (Proof: if (a,L) makes $\mathcal{A}$ manifestly symmorphic, so does the translation $(0,L^{-1})(a,L)$). It will be said <u>proper</u> symmorphic if $\mathcal{T}_A$ and $\mathcal{L}_A$ are both different from zero. If $\mathcal{A} = \mathcal{T}_A$ it is a <u>translational</u> algebra, if $\mathcal{A} = \mathcal{L}_A$, it is a <u>homogeneous</u> algebra.

Now suppose that $\mathcal{T}_A + \mathcal{L}_A \subset \mathcal{A}$. We denote by $\mathcal{M}_A$ a complementary subspace of $\mathcal{T}_A + \mathcal{L}_A$ in $\mathcal{A}$:

(III.7) $$\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$$

It is clear that any non-zero element m_i of $\mathcal{M}_A$ can be written as a sum of one element t_i of $\mathcal{T}$ and one element ℓ_i of $\mathcal{L}$

(III.8) $$m_i = t_i + \ell_i .$$

This decomposition is unique and non-trivial (i.e. neither t_i or ℓ_i can be zero). Moreover $t_i \notin \mathcal{T}_A$, $\ell_i \notin \mathcal{L}_A$. Let us denote by $\mathcal{T}'_A$ (resp. $\mathcal{L}'_A$) the subspace spanned by the t_i's (resp. ℓ_i's). It is easy to check the following properties :

(III.9) $$[\mathcal{L}_A + \mathcal{L}'_A , \mathcal{L}_A + \mathcal{L}'_A] \subseteq \mathcal{L}_A + \mathcal{L}'_A$$

(III.10) $$[\mathcal{L}_A + \mathcal{L}'_A , \mathcal{T}_A] \subseteq \mathcal{T}_A$$

(III.11) $$[\mathcal{L}_A , \mathcal{T}_A + \mathcal{T}'_A] \subseteq \mathcal{T}_A + \mathcal{T}'_A$$

that is $\mathcal{T}_A + (\mathcal{L}_A + \mathcal{L}'_A)$ and $(\mathcal{T}_A + \mathcal{T}'_A) + \mathcal{L}_A$ are symmorphic subalgebras with the same dimension as $\mathcal{A}$ since

(III.12) $$\dim \mathcal{T}'_A = \dim \mathcal{L}'_A = \dim \mathcal{M}_A$$

and

(III.13) $$\mathcal{T}_A \cap \mathcal{T}'_A = 0 , \qquad \mathcal{L}_A \cap \mathcal{L}'_A = 0$$

It follows that it is always possible to associate with each non-symmorphic subalgebra $\mathcal{A}$, two symmorphic ones with the same dimension as $\mathcal{A}$. These subalgebras will be said to be associated with $\mathcal{A}$. More precisely, $\mathcal{T}_A \dot{+} (\mathcal{L}_A + \mathcal{L}'_A)$ will be called <u>ℓ-associated</u> and $(\mathcal{T}_A + \mathcal{T}'_A) \dot{+} \mathcal{L}_A$ will be called <u>t-associated</u>.

We can sum up the above discussion in the following theorem

<u>Theorem III.1</u>

Given an inhomogeneous Lie algebra $\mathcal{P} = \mathcal{T} \dot{+} \mathcal{L}$, any Lie subalgebra $\mathcal{A}$ can be written in the form $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$. Every non-zero vector of $\mathcal{M}_A$, if any, can be written in a unique way as a sum of a vector of a subspace $\mathcal{L}'_A$ of $\mathcal{L}$ and a vector of a subspace $\mathcal{T}'_A$ of $\mathcal{T}$. Moreover, $\mathcal{T}_A + \mathcal{L}_A + \mathcal{L}'_A$ and $\mathcal{T}_A + \mathcal{T}'_A + \mathcal{L}_A$ are symmorphic algebra of the same dimension as $\mathcal{A}$.

Let us now prove the following properties .

Theorem III.2

The quotient spaces $\overline{\mathcal{L}'_A} = (\mathcal{L}_A + \mathcal{L}'_A)/\mathcal{L}_A$ and $\overline{\tau'_A} = (\tau_A + \tau'_A)/\tau_A$ are representation spaces for the subalgebra $\mathcal{L}_A$. These two representations are equivalent.

Let us first note that $\overline{\mathcal{L}'_A}$ (resp. $\overline{\tau'_A}$) can be identified, as vector spaces, with $\mathcal{L}'_A$ (resp. τ'_A) . Now, since $\mathcal{L}_A + \mathcal{L}_A$ is a subalgebra, one has :

$$[\mathcal{L}_A, \overline{\mathcal{L}'_A}] \subseteq \overline{\mathcal{L}'_A} \qquad \text{(III.14)}$$

Similarly, since $\tau_A + \tau'_A$ is an invariant subspace under $\mathcal{L}_A$

$$[\mathcal{L}_A, \overline{\tau'_A}] \subseteq \overline{\tau'_A} \ . \qquad \text{(III.15)}$$

Moreover, we already know from property (III.8) that there exists a one-to-one linear mapping from $\mathcal{M}_A$ to $\mathcal{L}'_A$ and one from $\mathcal{M}_A$ to τ'_A. Therefore, the two representations of $\mathcal{L}_A$ defined by (III.14) and (III.15) are equivalent.

Theorem III.3 .

Let $\mathcal{A} = \tau_A + \mathcal{L}_A + \mathcal{M}_A$ be a subalgebra of $\mathcal{P}$ and define the set $\mathcal{S}_A = \{t \in \tau \mid [\mathcal{L}_A, t] \subseteq \tau_A\}$.

i) If $\mathcal{A}$ is symmorphic, there exists $t_o \in \mathcal{S}_A$, $t_o \notin \tau_A$ such that t_o generates a translation automorphism mapping $\mathcal{A}$ on the manifestly symmorphic algebra $\tau_A + (\mathcal{L}_A + \mathcal{L}'_A)$.

ii) If $\mathcal{S}_A = \tau_A$, $\mathcal{A}$ is nonsymmorphic.

Proof : Obviously, $\tau_A \subseteq \mathcal{S}_A$.

We already noted at the beginning of the section that if $\mathcal{A}$ is symmorphic, there exists a translation automorphism which transforms it into a manifestly symmorphic subalgebra. Clearly, this transformation can be chosen in $\mathcal{S}_A$, the translational part of the stabilizer of $\tau_A + \mathcal{L}_A$. Let us denote by e^{t_o} this translation. The element $m = \ell' + t'$ of $\mathcal{M}$ is transformed as follows :

$$e^{t_o} m e^{-t_o} = m - [\ell', t_o] = \ell' + t' - [\ell', t_o] \ . \qquad \text{(III.16)}$$

If $\mathcal{A}$ is symmorphic, $t' - [\ell', t_o]$ must belong to τ_A . Therefore, t_o cannot belong to τ_A since, according to Eq. (III.10)

$$t_o \in \tau_A \Longrightarrow [\ell', t_o] \in \tau_A \Rightarrow t' - [\ell', t_o] \notin \tau_A \ .$$

The theorem follows.

Theorem III.4 .

If $\mathcal{L}$ is semi-simple, every subalgebra of the form $\mathcal{A} = \mathcal{L}_A + \mathcal{M}_A$ with $\mathcal{L}_A + \mathcal{L}'_A = \mathcal{L}$ is conjugate to $\mathcal{L}$.

Proof : Let σ be the isomorphism between $\mathcal{L}$ and $\mathcal{A}$ defined by :

$$\sigma(\ell) = \ell \ , \quad \ell \in \mathcal{L}_A$$
$$\sigma(\ell') = \ell' + t' \ , \quad \ell' \in \mathcal{L}'_A \text{ and } \ell' + t' \in \mathcal{M}_A \qquad \text{(III.17)}$$

σ can be extended to an automorphism τ of $\mathcal{P}$ by defining its action on τ

$$\tau(t) = t \ , \quad t \in \tau. \qquad \text{(III.18)}$$

The action of τ on $\mathcal{L}$ can be written .

$$\tau(\ell) = \ell + t(\ell) \ , \quad \ell \in \mathcal{L} \ , \quad t(\ell) \in \tau \ . \qquad \text{(III.19)}$$

Let ℓ_1, ℓ_2 be two elements of $\mathcal{L}$; the relation .

$$\tau([\ell_1, \ell_2]) = [\tau(\ell_1), \tau(\ell_2)] \qquad \text{(III.20)}$$

(...21) $\tau([t_1,t_2]) = [t_1,\tau(t_2)] - [t_2,\tau(t_1)]$

this relation is a 1-cocycle equation. As $\mathcal{L}$ is semi-simple every 1-cocycle is a 1-coboundary. Consequently, there exists $t_\tau \in \mathcal{T}$ such that

(III.22) $\tau(t) = [t,t_\tau] \quad , \quad t \in \mathcal{L}$

then

(III.23) $\tau(t) = t + [t,t_\tau]$.

This last relation and the fact that τ acts identically on $\mathcal{T}$ prove that τ is a translation.

We are interested in finding all the maximal subalgebras of $\mathcal{P}$, i.e. all algebras $\mathcal{A}$ such that $\mathcal{A} \subset \mathcal{P}$ and $\nexists\, \mathcal{B}$ with $\mathcal{A} \subset \mathcal{B} \subset \mathcal{P}$.

It follows that $\mathcal{T}'_A = 0$ and $\mathcal{M}_A = 0$. $\mathcal{A}$ is a symmorphic algebra. The maximality condition for $\mathcal{A}$ implies

(III.24) $\mathcal{A} = \mathcal{T} + \mathcal{L}_A$

If we want to find $\mathcal{A}$ up to a conjugation, it is clear, due to the semi-direct product structure, that $\mathcal{L}_A$ must be chosen up to a conjugation in $\mathcal{L}$.

2) $\mathcal{T}_A \subset \mathcal{T}$

In that case, $\mathcal{L}_A + \mathcal{L}'_A = \mathcal{L}$. Indeed, $\mathcal{L}_A + \mathcal{L}'_A \subset \mathcal{L}$ would imply :

(III.25) $\mathcal{A} \subset \mathcal{L}_A + \mathcal{L}'_A + \mathcal{T} \subset \mathcal{P}$

and $\mathcal{A}$ could not be maximal.

By a similar discussion and according to Eq. (III.10), $\mathcal{T}_A$ must be a maximal invariant subspace under $\mathcal{L}$, i.e. a maximal ideal in $\mathcal{P}$.

In the case $\mathcal{L}'_A = 0$, we have : $\mathcal{L}_A = \mathcal{L}$, and every maximal invariant subspace $\mathcal{T}_A$ of $\mathcal{L}$ provides us with a maximal symmorphic subalgebra, namely :

(III.26) $\mathcal{A} = \mathcal{T}_A + \mathcal{L}$

If we choose $\mathcal{T}_A$ up to a conjugation in $\mathcal{L}$, then we get $\mathcal{A}$ up to a conjugation in $\mathcal{P}$.

All these properties are stated in the following theorem :

Theorem III.5 :

If $\mathcal{A}$ is a maximal subalgebra of $\mathcal{P}$, it belongs to one of the three categories

a) $\mathcal{A} = \mathcal{T} + \mathcal{L}_A$ ($\mathcal{A}$ symmorphic), where $\mathcal{L}_A$ is a maximal subalgebra of $\mathcal{L}$.

b) $\mathcal{A} = \mathcal{T}_A + \mathcal{L}$ ($\mathcal{A}$ symmorphic), where $\mathcal{T}_A$ is a maximal invariant subspace under $\mathcal{L}$.

c) $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$ ($\mathcal{A}$ non symmorphic) with the following properties :

i) $\mathcal{L}_A + \mathcal{L}'_A = \mathcal{L}$

ii) $\mathcal{T}_A$ is a maximal invariant subspace under $\mathcal{L}$.

iii) $\mathcal{T}_A + \mathcal{T}'_A$ is an invariant subspace under $\mathcal{L}_A$.

Remark : Properties i) and ii) can be put together in the following way : the $\mathcal{L}$-associated algebra of $\mathcal{A}$ is maximal.

A variant of this theorem can be given when $\mathcal{L}$ acts irreducibly on $\mathcal{T}$.

Theorem III.6

If $\underline{L}$ acts irreducibly on $\mathcal{T}$, a maximal subalgebra $\mathcal{A}$ of $\mathcal{P}$ has one of the following forms :

a) $\mathcal{A} = \mathcal{T} + \mathcal{L}_A$, where $\mathcal{L}_A$ is a maximal subalgebra of $\mathcal{L}$.

b) $\mathcal{A} = \mathcal{L}$.

c) $\mathcal{A} = \mathcal{L}_A + \mathcal{M}_A$, with $\mathcal{L}_A + \mathcal{L}'_A = \mathcal{L}$ and $\mathcal{A}$ is isomorphic to $\mathcal{L}$.

Combining this last theorem with Theorem III.4, we obtain

Theorem III.7

If $\underline{L}$ is semi-simple and acts irreducibly on $\mathcal{T}$, every maximal subalgebra of $\mathcal{P}$ is symmorphic.

The reader is invited to prove the following result.

Theorem III.8

If $\underline{L}$ is of dimension one and acts irreducibly on $\mathcal{T}$, the only maximal subalgebras of $\mathcal{P}$ are $\mathcal{T}$ and $\mathcal{L}$.

C - PARTICULAR CASE OF A DIRECT PRODUCT

A particular case of inhomogeneous Lie group is the one where the semi-direct product $\underline{T} \square \underline{L}$ reduces to the direct product $\underline{T} \times \underline{L}$. The relations (III.2) become :

$$\text{(III.27)} \qquad \begin{cases} [\mathcal{L},\mathcal{L}] & \subseteq & \mathcal{L} \\ [\mathcal{L},\mathcal{T}] & = & 0 \\ [\mathcal{T},\mathcal{T}] & = & 0 \end{cases}$$

The previous results can still be used for classifying the maximal subalgebras of $\mathcal{P}$. However, some more informations can be given about these subalgebras.

Let us first remark that, if $\dim(\mathcal{T}) > 1$, the set of maximal invariant subspaces of $\mathcal{T}$ under $\mathcal{L}$ is the set of hyperplanes $\mathcal{T}_A$ of $\mathcal{T}$ (defined by $\dim(\mathcal{T}_A) = \dim(\mathcal{T}) - 1$). According to relation (III.16) any subspace $\mathcal{T}_A + \mathcal{L}$, where $\mathcal{T}_A$ is a given hyperplane, provides us with a maximal subalgebra. Two distinct hyperplanes correspond to non conjugate algebras.

Moreover, let us note that, for a nonsymmorphic maximal subalgebra $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$, a non-zero element of $\mathcal{L}'_A$ cannot belong to the derived algebra $\mathcal{L}' = [\mathcal{L},\mathcal{L}]$, for $\mathcal{L}'$ is also equal to $[\mathcal{L}_A + \mathcal{M}_A, \mathcal{L}_A + \mathcal{M}_A]$. As a consequence :

i) if $\mathcal{L}' = \mathcal{L}$, every maximal algebra is symmorphic.

ii) if $\mathcal{L}' \neq \mathcal{L}$, $\mathcal{M}_A$ is necessarily of dimension one and is generated by an element $t_o + t_o$ with $t_o \in \mathcal{L} - \mathcal{L}'$ and $t_o \in \mathcal{T}$ such that $\mathcal{T}_A + [t_o] = \mathcal{T}$ ($[t_o]$ is the one-dimensional vector space generated by t_o). It follows that : $\dim \mathcal{L}_A = \dim \mathcal{L} - 1$.

Two trivial cases are those corresponding to $\dim \mathcal{T} = 1$ or $\dim \mathcal{L} = 1$:

a) If $\dim \mathcal{T} = 1$ (and $\dim \mathcal{L} > 1$), maximal subalgebras are

i) $\mathcal{L}$.

ii) $\mathcal{T} + \mathcal{L}_A$, with $\mathcal{L}_A$ maximal in $\mathcal{L}$.

iii) when $\mathcal{L}' \neq \mathcal{L}$, $\mathcal{L}_A + \mathcal{M}_A$ with $\mathcal{M}_A = [t_o + t]$, $t_o \in \mathcal{L} - \mathcal{L}'$, $[t] = \mathcal{T}$ and $\dim \mathcal{L}_A = \dim \mathcal{L} - 1$.

b) If $\dim \mathcal{L} = 1$ (and $\dim \mathcal{T} > 1$), maximal subalgebras are :

i) $\mathcal{T}$.

ii) $\mathcal{T}_A + \mathcal{L}$, with $\dim \mathcal{T}_A = \dim \mathcal{T} - 1$.

iii) $\mathcal{T}_A + \mathcal{M}_A$, with $\mathcal{M}_A = [t + t_o]$ with $[t] = \mathcal{L}$ and $\mathcal{T}_A + [t_o] = \mathcal{T}$.

We can then add the following corollary to the results of § B :

Theorem III.9

If $[\mathcal{L},\tau] = 0$, any maximal subalgebra $\mathcal{A}$ of $\mathcal{P}$ belongs to one of the three types

a) $\mathcal{A} = \tau + \mathcal{L}_A$, with $\mathcal{L}_A$ maximal in $\mathcal{L}$.

b) $\mathcal{A} = \tau_A + \mathcal{L}$, with $\dim \tau_A = \dim \tau - 1$.

c) $\mathcal{A} = \tau_A + \mathcal{L}_A + \mathcal{N}_A$ (only if $[\mathcal{L},\mathcal{L}] \neq \mathcal{L}$) and

i) $\dim \tau_A = \dim \tau - 1$.

ii) $\dim \mathcal{L}_A = \dim \mathcal{L} - 1$.

All the above theorems will be used in the next sections in order to build $\mathcal{H}(\mathcal{P})$. Let us give some idea about the way of using them. According to Theorem III.2 , it will be necessary to know the Lorentz $\mathcal{H}(L)$ and the action of every Lorentz subgroup on space-time. This will be studied in section IV. The property of the Lorentz group to be simple is of great advantage. In particular, the Lie algebra of its automorphisms is the Lorentz Lie algebra itself (see Appendix A) and permits us to use Theorem III.4 .

Maximal subalgebras of $\mathcal{P}$ and E(3) , E(2,1) are readily obtained with the aid of Theorems III.5 , 6 , 7 . All of them will be shown to be symmorphic ; therefore, the same theorems will be used for the maximal subalgebras of these maximal subalgebras.

The symmorphic algebras are easily derived from results of section IV. Every couple of symmorphic algebras $\mathcal{A}$ and $\mathcal{B}$ of same dimension, satisfying

(III.28) $\mathcal{L}_A \subset \mathcal{L}_B$ and $\tau_A \subset \tau_B$

could be the associated algebras of some nonsymmorphic subalgebras, provided conditions of Theorem III.2 are satisfied. It is clear that we have to verify that the subalgebras obtained in this way are not conjugate to symmorphic ones. This is done with the aid of Theorem III.3 .

IV - CONNECTED SUBGROUPS OF THE TRANSLATION GROUP T(4) AND OF THE LORENTZ GROUP L

To obtain the set $\mathcal{H}(P)$ of an inhomogeneous Lie group $P = T \square L$, it is necessary to study separatly the subgroups of L and those of T. We already know from results of Section II that the set $\mathcal{H}(L)$ is a subset of $\mathcal{H}(\mathcal{L})$.
This property follows from the fact that two elements of the group L are conjugate in P if and only if they are conjugate in L.
A similar property holds for the group T. Two elements of T are conjugate in P if and only if they are conjugate under a homogeneous transformation (the proof is left to the reader).

A) CONNECTED SUBGROUPS OF T(4)

The set $\mathcal{H}(T(4))$ is readily obtained. It is given without proof in Table IV.1.

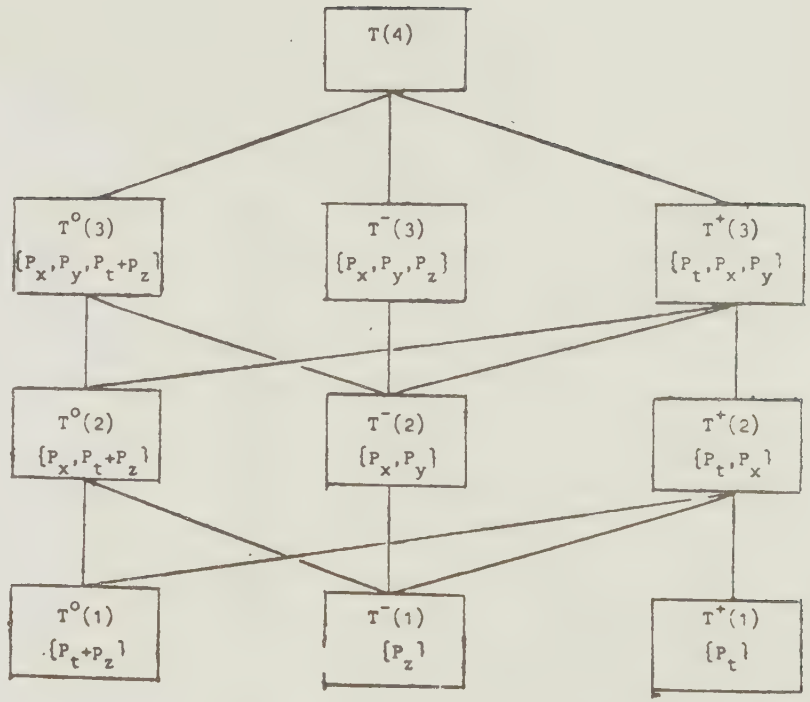

Table IV.1 - Connected subgroups of T(4)

The symbols + , - , o , characterize the type of one, two and three-dimensional subspaces under consideration. An upscript o denotes a subspace tangent to the light cone, an upscript + is used if the corresponding subspace contains one time like vector and the minus sign is associated with a space-like subspace.

B) LATTICE OF THE CONNECTED SUBGROUPS OF L

a) Maximal connected subgroups of L

Dynkin [42'] has given some important theorems on the maximal connected subgroups of simple complex Lie groups. According to these theorems, the maximal connected subgroups of SL(n,c) are of the two following types :

i) The groups of all matrices of $SL(n,c)$ which leave invariant a p-dimensional subspace of C^n $(0 < p < n)$. In the case of the Lorentz group $(n = 2)$, there is only one group of this type corresponding to $p = 1$, namely the group of matrices

$$\begin{vmatrix} a & b \\ 0 & \frac{1}{a} \end{vmatrix} \qquad \text{(IV.1)}$$

This group is four-dimensional and will be denoted $S(2)$ because it is isomorphic to the similitude group in two dimensions.

ii) The real forms of $SL(n,c)$. They have been classified by E. Cartan [38]. For $n = 2$, they are $SU(2)$ and $SU(1,1)$ (isomorphic to $SL(2,R)$; both are simple.

The theorem of Borel-Morozov [44] assures the existence, in a semi-simple complex Lie algebra $\mathcal{G}$, of a solvable maximal subalgebra $\mathcal{B}$, which is called the Borel algebra. This subalgebra is unique up to a conjugation. Moreover, every solvable subalgebra of $\mathcal{G}$ is conjugate to some subalgebra of $\mathcal{B}$.

It readily follows that $\underline{S}(2)$ the Borel algebra of $\mathcal{L}$ contains all solvable subalgebras of $\mathcal{L}$ and that every subalgebra of $\underline{S}(2)$ is solvable.

Let us note that the subalgebras of $\underline{SU}(2)$ and $\underline{SU}(1,1)$ are necessarily of dimension less than three ; therefore they are solvable and belong to $\mathcal{N}\left(\underline{S}(2)\right)$. Consequently, all subalgebras of $\underline{SL}(2,C)$, except $\underline{SU}(2)$ and $\underline{SU}(1,1)$, are subalgebras of $\underline{S}(2)$.

b) Connected subgroups of $S(2)$

It can be shown that two subgroups of $S(2)$ which are conjugate under a Lorentz transformation are conjugate under a transformation of $S(2)$. This does not mean that this conjugation property is true for the elements themselves, but it is easy to prove that if $S_1, S_2 \in S(2)$ are Lorentz

conjugate, either (S_1, S_2) or (S_1, S_2^{-1}) are $S(2)$ conjugate [12].

Consequence : The p.o.s. $\mathcal{N}(S(2))$ is a subset of $\mathcal{N}(\mathcal{L})$.

Now we note that $S(2)$ is itself an inhomogeneous group since it is the semi-direct product of the Abelian group $A(2)$ generated by A_x and A_y (the so-called horyspheric group [45]) by the group $SO(2) \times L(1,1)$ generated by J_z and K_z. With respect to this structure, we can classify its subgroups in symmorphic and nonsymmorphic ones. (Of course, they are symmorphic with respect to the Poincaré group).

According to Theorem III.6, the maximal connected subgroups of $S(2)$ are ${}^{L}E(2) = A(2) \square SO(2)$ (isomorphic to the Euclidean group in two dimensions generated by J_z, A_x, A_y), $Th(2) = A(2) \square L(1,1)$ (isomorphic to the Thales group generated by K_z, A_x, A_y), $S_f(2) = A(2) \square F_f$ (generated by J_f, A_x, A_y) and $SO(2) \times L(1,1)$. Every maximal nonsymmorphic subalgebra, if any, must be isomorphic to it and, therefore, abelian. From Theorem III.2, and due to the action of $SO(2) \times L(1,1)$ on $A(2)$, there is no such algebra.

We are led to find the maximal subalgebras of the maximal subalgebras of $S(2)$. According to Theorem III.8, all of them are symmorphic. They are :

maximal subalgebras of ${}^{L}E(2)$: $SO(2)$ and $A(2)$;
of $Th(2)$: $S(1)$ and $A(2)$;
of $S_f(2)$: F_f and $A(2)$;
of $SO(2) \times L(1,1)$: $SO(2)$, $L(1,1)$ and F_f.

All results are summarized in Table IV.2.

(12) The proof is left to the reader. It is based on simple matrix calculus.

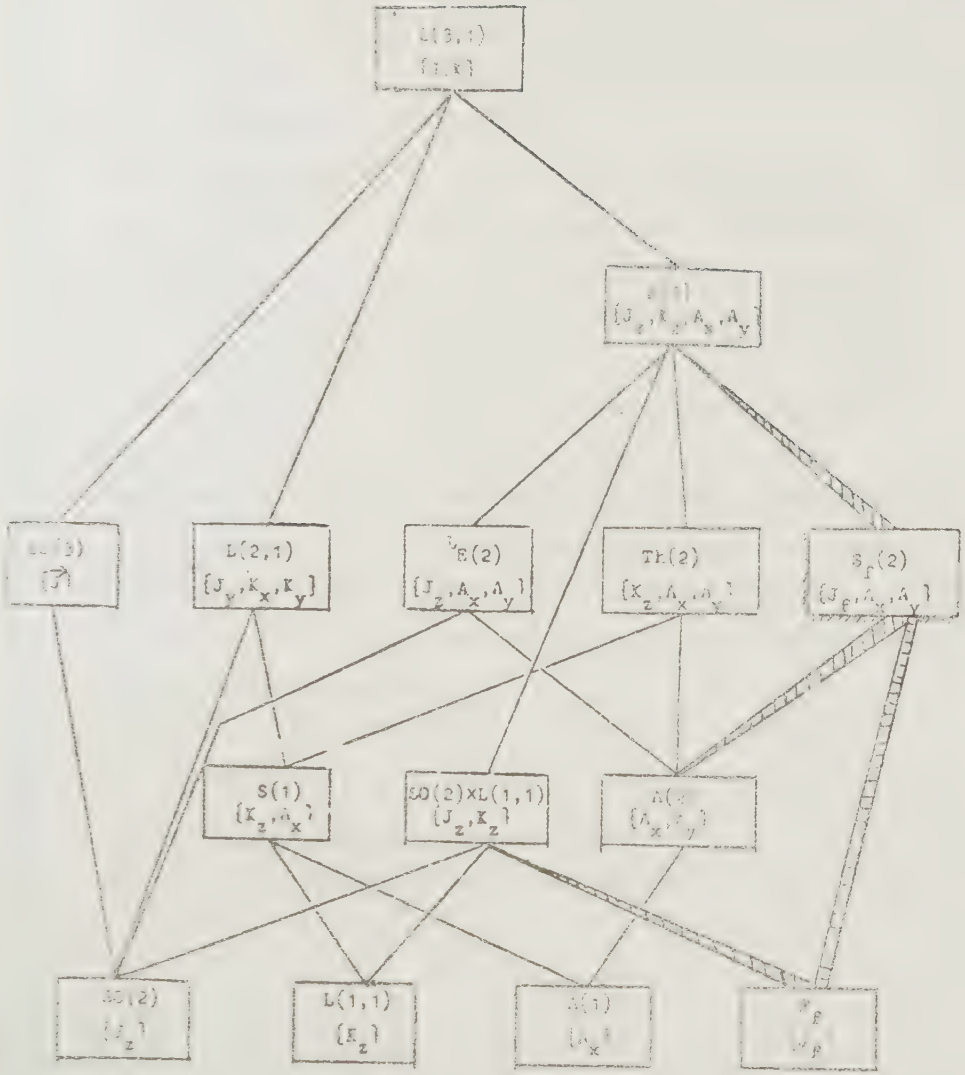

Table IV.2 - Connected subgroups of the connected Lorentz group L(3,1)

C) STABILIZERS OF THE CONNECTED SUBGROUPS OF THE LORENTZ GROUP

We are interested in finding the stabilizers of all subalgebras of the Lorentz Lie algebra. The following lemma will help us to derive all of them.

Lemma

Let $\mathcal{L}_B$ be a subalgebra of $\mathcal{L}$. The subgroup of $\underline{P} = \underline{T} + \underline{L}$ which stabilizes $\mathcal{L}_B$ has a symmorphic Lie algebra.

Proof : Suppose the Lie algebra $\mathcal{A}$ of the stabilizer to be nonsymmorphic. One has

$$[\mathcal{L}_B,\mathcal{A}] = [\mathcal{L}_B,\tau_A + \mathcal{L}_A + M_A] \subseteq \mathcal{L}_B \quad \text{with } M_A \neq 0 \tag{IV.2}$$

which implies

$$[\mathcal{L}_B,M_A] \cap \tau = [\mathcal{L}_B,\tau'_A] = 0 . \tag{IV.3}$$

Therefore, τ'_A would belong to $\mathcal{A}$, in contradiction with the nonsymmorphic character of $\mathcal{A}$.

Note that (IV.2) implies also

$$\mathcal{L}_B \subseteq \mathcal{L}_A \tag{IV.4}$$

and

$$[\mathcal{L}_B,\tau_A] = 0 . \tag{IV.5}$$

The lemma, together with Eqs (IV.4) and (IV.5) will be used in our derivation of the stability subgroups of Lorentz subalgebras. In fact, Eq (I.5) provides us with the Lorentz stability subgroup and Eq (IV.5) with the translational part. The results are gathered in Table IV-3 for Lorentz subalgebras of dimension larger than one since those corresponding to one-dimensional algebras are already in Section II .

Lorentz connected Subgroups		Invariant Subspaces of T(4) with their generators (14)		
Dimension / Subgroups and their generators		Dimension 1	Dimension 2	Dimension 3
4	$S(2);\{J_z, K_z, A_x, A_y\}$	$T^o(1);\{P_t - P_z\}$		$T^o(3);\{P_x, P_y, P_t - P_z\}$
3	$SO(3);\{J_x, J_y, J_z\}$	$T^+(1);\{P_t\}$		$T^-(3);\{P_x, P_y, P_z\}_{inv}$
	$L(2,1);\{J_y, K_x, K_z\}$	$T^-(1);\{P_y\}$		$T^+(3);\{P_x, P_z, P_t\}_{inv}$
	$^LE(2);\{J_z, A_x, A_y\}$	$T^o(1);\{P_t - P_z\}$		$T^o(3);\{P_x, P_y, P_t - P_z\}$
	$Th(2);\{K_z, A_x, A_y\}$	$T^o(1);\{P_t - P_z\}$	$T^o(2);\{\alpha P_x + \beta P_t, P_t - P_z\}$	$T^o(3);\{P_x, P_y, P_t - P_z\}$
	$S_f(2);\{J_f, A_x, A_y\}$	$T^o(1);\{P_t - P_z\}$		$T^o(3);\{P_x, P_y, P_t - P_z\}$
2	$S(1);\{K_z, A_x\}$	$T^o(1);\{P_t - P_z\}$ $T^-(1);\{P_y\}$	$T^o(2);\{\alpha P_x + \beta P_y, P_t - P_z\}$	$T^o(3);\{P_x, P_y, P_t - P_z\}$ $T^+(3);\{P_x, P_z, P_t\}$
	$SO(2)XL(1,1);\{J_z, K_z\}$	$T^o(1);\{P_t \pm P_z\}$	$T^+(2);\{P_x, P_t\}$ $T^-(2);\{P_z, P_y\}_{inv}$	$T^o(3);\{P_x, P_y, P_t \pm P_z\}$
	$A(2);\{A_x, A_y\}$	$T^o(1);\{P_t - P_z\}$	$T^o(2);\{\alpha P_x + \beta P_y, P_t - P_z\}$	$T^o(3);\{P_x, P_y, P_t - P_z\}$
1	$SO(2);\{J_z\}$	$T^o(1);\{P_t \pm P_z\}$ $T^+(1);\{\alpha P_t - \beta P_z\}$ $\alpha^2 > \beta^2$ $T^-(1);\{\alpha P_t - \beta P_z\}$ $\alpha^2 < \beta^2$	$T^+(2);\{P_x, P_t\}$ $T^-(2);\{P_x, P_y\}_{inv}$	$T^o(3);\{P_x, P_y, P_t \pm P_z\}$ $T^-(3);\{P_x, P_y, \beta P_t + \alpha P_z\}$ $\alpha^2 > \beta^2$ $T^+(3);\{P_x, P_y, \beta P_t + \alpha P_z\}$ $\alpha^2 < \beta^2$
	$L(1,1);\{K_z\}$	$T^o(1);\{P_t \pm P_z\}$ $T^-(1);\{\alpha P_x + \beta P_y\}$	$T^+(2);\{P_t, P_z\}$ $T^o(2);\{\alpha P_x + \beta P_y, P_t - P_z\}$ $T^-(2);\{P_x, P_y\}$	$T^o(3);\{P_x, P_y, P_t \pm P_z\}$ $T^+(3);\{\alpha P_x + \beta P_y, P_z, P_t\}$
	$A(1);\{A_x\}$	$T^o(1);\{P_t - P_z\}$ $T^-(1);\{P_y, \alpha(P_t - P_z)\}$	$T^o(2);\{\alpha P_x + \beta P_y, P_t - P_z\}$	$T^o(3);\{P_x, P_y, P_t - P_z\}$ $T^+(3);\{P_x, P_z, P_t\}$
	$F_p;\{J_p\}$	$T^o(1);\{P_t \pm P_z\}$	$T^+(2);\{P_z, P_t\}$ $T^-(2);\{P_x, P_y\}_{inv}$	$T^o(3);\{P_x, P_y, P_t \pm P_z\}$

Table IV-4 : Invariant Subspaces of T(4) under Lorentz connected Subgroups

(14) irreducibility of the subspace is mentionned for dimension 2 and 3 .

Lorentz connected subgroups		stabilizaters and their generators (13)
dimension	subgroups and generators	
4	$S(2) : \{J_z, K_z, A_x, A_y\}$	$S(2) : \{J_z, K_z, A_x, A_y\}$
3	$SO(3) : \{J_x, J_y, J_z\}$	$T^+(1) \times SO(3) : \{P_t, J_x, J_y, J_z\}$
	$L(2,1) : \{J_y, K_x, K_z\}$	$[T^-(1) \times L(2,1)] \square Z_2 : \{P_y, J_y, K_x, K_z\}$
	$^LE(2) : \{J_z, A_x, A_y\}$	$T^o(1) \times S(2) : \{P_t - P_z, J_z, K_z, A_x, A_y\}$
	$Th(2) : \{K_z, A_x, A_y\}$	$S(2) : \{J_z, K_x, A_x, A_y\}$
	$S_f(2) : \{J_f, A_x, A_y\}$	$S(2) : \{J_z, K_z, A_x, A_y\}$
2	$S(1) : \{K_z, A_x\}$	$[T^-(1) \times S(1)] \square Z_2 : \{P_y, K_z, A_x\}$
	$SO(2) \times L(1,1) : \{J_z, K_z\}$	$[SO(2) \times L(1,1)] \square Z_2 : \{J_z, K_z\}$
	$A(2) : \{A_x, A_y\}$	$T^o(1) \times S(2) : \{P_t - P_z, J_z, K_z, A_x, A_y\}$

Table IV-3 : Stabilizaters of the connected subgroups of the Lorentz group

We have found convenient to build a table where all invariant subspaces of the Minkowski space are given for each connected subgroup of the Lorentz subgroup. These results are taken from Shaw [27] . Tables IV-3 and IV-4 will appear to be very useful for constructing the complete set $\mathcal{N}(\underline{P})$.

(13) The discrete transformation corresponds to a rotation of π around the y^- axis.

V - CONNECTED SUBGROUPS OF THE EUCLIDEAN GROUP E(3) AND THE PSEUDO-EUCLIDEAN GROUP E(2,1)

The importance of the Euclidean group E(3) in Physics would be sufficient to justify a separate investigation of its subgroups. $\mathcal{T}(E(3))$ has a very nice property (in common with that of the Lorentz group), namely to be a subset of $\mathcal{T}(\underline{P})$. In other words, two subgroups of E(3) are conjugate in E(3) if and only if they are conjugate in $\underline{P}$. Surprisingly, this property is not true for the pseudo-Euclidean group E(2,1) -the Poincaré group of a 3-dimensional space-time- but we found more convenient to investigate it in the same chapter, because $\mathcal{T}(E(2,1))$ is "almost" a subset $\mathcal{T}(\underline{P})$.

A - SUBGROUPS OF E(3)

The group E(3) is the semi-direct product $T^{-}(3) \square SO(3)$ generated by P_x, P_y, P_z, J_x, J_y, J_z.

Theorem V.1

Two elements of E(3) are conjugate in P if and only if they are conjugate in E(3). The E(3) conjugacy transformation only depends on the $\underline{P}$-conjugacy transformation.

(Note the similarity between this theorem and the lemma of Section II for the Lorentz group).

This property is a consequence of the following theorem.

Theorem V.2

Two unitary nxn - matrices U_1 and U_2 which are conjugate under a non-singular linear transformation Λ are conjugate under the unitary argument of Λ.

The proof is based on the well-known polar decomposition [46] of a nonsingular matrix Λ

(V.1) $$\Lambda = U H = H' U$$

which is unique under the conditions that H and H' are Hermitian positive and U is unitary (the unitary argument of Λ). One has :

(V.2) $$\Lambda^{+}\Lambda = H^2 \quad , \quad \Lambda\Lambda^{+} = H'^2 .$$

Let U_1 and U_2 be two unitary matrices which are conjugate under a nonsingular matrix Λ. Since $\Lambda U_1 \Lambda^{-1}$ is unitary, one can write :

(V.3) $$U_1^{+} \Lambda^{+}\Lambda U_1 = \Lambda^{+}\Lambda ;$$

or, according to (V.2)

(V.4) $$U_1^{+} H^2 U_1 = H^2 .$$

Now, the other polar product $H U_1$ can be written $U_1 H_1$. Therefore, (Eq. V.4) becomes

(V.5) $$(U_1^{+}H)(HU_1) = H_1 U_1^{+} U_1 H_1 = H_1^2 = H^2 .$$

Since H (resp. H_1) is the positive square root of H^2 (resp. H_1^2), one obtains

(V.6) $$H_1 = H \qquad \text{and} \qquad HU_1 = U_1 H .$$

Consequently,

(V.7) $$U_2 = \Lambda U_1 \Lambda^{-1} = UHU_1 H^{-1} U^{-1} = UU_1 U^{-1}$$

which proves Theorem V.2 .

It follows, from this theorem, that two rotations are conjugate under a Lorentz transformation if and only if they are conjugate under a rotation. Due to the uniqueness of the unitary argument in the polar decomposition, the rotation transformation associated with a given Lorentz transformation is unique.

Let us now prove Theorem V.1 . Let (a_1, R_1) and (a_2, R_2) be

The Galilei transformations are conjugate under the Poincaré transformation (a,L) :

$$(V.8) \qquad (\vec{a}_2,R_2) = (a,L)(\vec{a}_1,R_1)(a,L)^{-1} = (a+L\vec{a}_1-LR_1L^{-1}a, LR_1L^{-1}) .$$

According to the above results L can be written as rotation-boost or boost-rotation product

$$(V.9) \qquad L = RB = B'R$$

with B commuting with R_1 . Therefore :

$$(V.10) \qquad (\vec{a}_2,R_2) = (a+RB\vec{a}_1-RR_1R^{-1}a \ , \ RR_1R^{-1}) .$$

The time component of $\vec{a}_2$ must be zero which implies, since the time-component of $a - RR_1R^{-1}a$ identically vanishes, that the boost B belongs to the stabilizer of vector $\vec{a}_1$. It readily follows that :

$$(V.11) \qquad (\vec{a}_2,R_2) = (\vec{a},R)(\vec{a}_1,R_1)(\vec{a},R)^{-1}$$

where $\vec{a}$ is the space part of 4-vector a , which proves our Theorem V.1 .

B - CONNECTED SUBGROUPS OF E(3)

We are going to make use of our general theorems of Section III to build the lattice of $E(3)$, which is already known to be a sublattice of $Lat(\mathcal{P})$.

From Theorems III.6 and III.7 , we readily get the maximal symmorphic subgroups of $E(3)$. They are [16]

$$Ar^-(2) = T^-(3) \square SO(2) = T^-(1) \times E(2) ,$$

generated by J_z , P_x , P_y , P_z ;

$E(3)$, generated by J_x , J_y , J_z .

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(15) We will call Aristotle groups $Ar^{\pm}(p,q)$ the direct product $T^{\pm}(1) \times E(p,q)$. The Aristotle group $Ar^+(3,1)$ is the largest subgroup common to Galilei and Poincaré groups.

The only subalgebra of $SO(3)$ is $SO(2)$. According to Theorem III.5 , the symmorphic maximal subalgebras of $Ar^-(2)$ are $T^-(3)$, $E(2)$ and $T^-(1) \times SO(2)$. The other symmorphic subalgebras are easily obtained. They are $SO(2)$, $T^-(2)$, $T^-(1)$. The non-symmorphic subalgebras are $T^-(2) \square H^+_\omega$ $(\omega \neq 0)$, generated by $\omega J_z+P_z, P_x, P_y$ (associated algebras $T^-(3)$ and $E(2)$). and H^+_ω $(\omega \neq 0)$ generated by ωJ_z+P_z (associated algebras $T^-(1)$ and $SO(2)$).

$\mathcal{M}(E(3))$ is described in Table V.1 , page V.5 .

C - SUBGROUPS OF $\hat{E}(2,1)$

We denote by $\hat{E}(2,1)$ the subgroup of $\underline{P}$, the connected part of which is $T^+(3) \square L(2,1) = E(2,1)$. Its Lie algebra is spanned by $P_t, P_x, P_y, J_z, K_x, K_y$. It is a two-sheeted group containing as an element the rotation of angle π around the y-axis[17] . We now have the following theorem .

Theorem V.3

Two elements of $\hat{E}(2,1)$ are conjugate in $\underline{P}$ if and only if they are conjugate in $\hat{E}(2,1)$. The $\hat{E}(2,1)$-transformation only depends on the $\underline{P}$-transformation.

The proof of this theorem is quite analogous to that of Theorem V.1 . To prove it, we need some counterpart of the polar decomposition where unitarity and Hermitianness are respectively replaced by pseudo-unitarity and pseudo-Hermitianness. None property of this kind is mentioned in literature. However it is not very difficult to investigate by

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(16) It is then isomorphic to $T^+(3) \square SO(2,1)$.

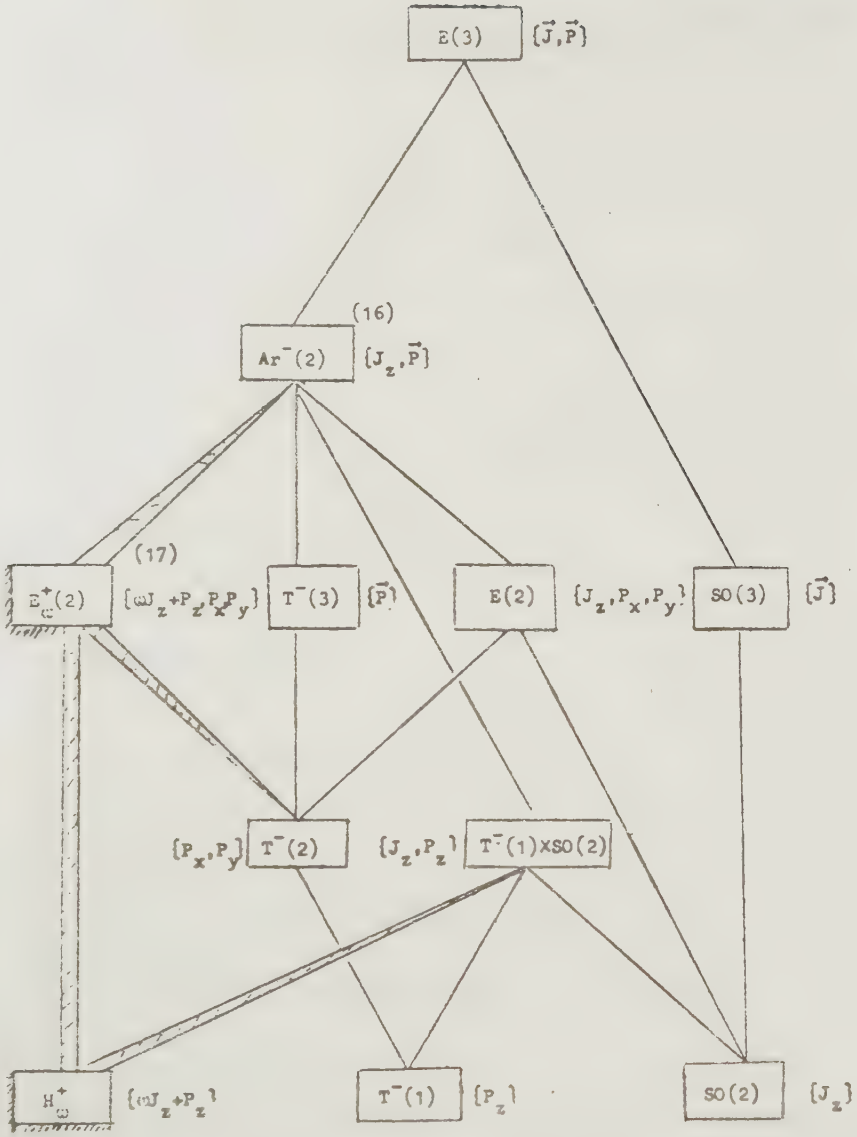


TABLE V.1 - Connected Subgroups of E(3)

(16) One has $Ar^-(2) = T^-(3) \square SO(2)$

(17) $E^+_\omega(2) = T^-(2) \supset H^+_\omega$ is isomorphic to E(2) .

direct matrix calculus the case of 2x2 matrices [19].

Let us define $\hat{SU}(1,1)$ as the subgroup of SL(2,C) whose elements $\hat{U}$ satisfy

$$\hat{U}\, g\, \hat{U}^+ = \varepsilon g \quad ,$$

where $\varepsilon = \pm 1$ and g is the matrix :

$$g = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} .$$

This is the group which leaves invariant the z-direction in space-time. Any element $\hat{U}$ is of the form :

$$\hat{U} = \begin{vmatrix} a & b \\ \varepsilon\bar{b} & \varepsilon\bar{a} \end{vmatrix} ,$$

with $|a|^2 - |b|^2 = \varepsilon$. It readily follows that $\hat{U}$ is either an element of $SU(1,1)$ or the product of $i\sigma_2 = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$ by some element of $SU(1,1)$. Due to the connectedness of $SU(1,1)$, it is clear that $\hat{SU}(1,1)$ is two-sheeted. The matrix $i\sigma_2$ corresponds to a rotation of angle π around the y-axis. The quotient space $SL(2,C)/\hat{SU}(1,1)$ is isomorphic to a hyperboloid of imaginary mass where two points which are symmetric with respect to the y-axis must be identified.

By a simple, but tedious, direct matrix calculation, the following theorem can be proved.

Theorem V.4

Any element Λ of $SL(2,C)$ can be written in the form

$$\Lambda = \hat{U} H ,$$

where $\hat{U} \in \hat{SU}(1,1)$ and H is pseudo-hermitian, i.e.

$$Hg = gH^+ .$$

(19) The authors are grateful to Dr. G. Rideau for many fruitful discussions on this point.

This decomposition is not generally unique.

Remark : It is equivalent to deal with any conjugate subgroup of $\hat{E}(2,1)$ in $\underline{P}$. It will be more convenient to use the one-generated by $P_t, P_x, P_z, J_y, K_x, K_z$. The rotation of angle π is the one performed around the z-axis.

D - CONNECTED SUBGROUPS OF E(2,1)

We first intend to derive, in this subsection, the p.o.s. $\mathcal{N}(E(2,1))$ all connected subgroups of E(2,1). Once $\mathcal{N}(E(2,1))$ will be obtained, it will be necessary to identify those subgroups which are conjugate in $\underline{P}$, that is, according to Theorem V.3, those which are conjugate in $\hat{E}(2,1)$. Since E(2,1) is of index two in $\hat{E}(2,1)$, we will have to identify those subgroups which are conjugate under a rotation of angle π around the z-axis.

E(2,1) is the semi-direct product $T^+(3) \square L(2,1)$ and the semi-simple group L(2,1) acts irreducibly on $T^+(3)$. Therefore Theorems III.6 and III.7 readily provide us with its maximal subalgebras. They are all symmorphic.:

a) $T^+(3) \square S(1)$ generated by P_t, P_x, P_z, A_x, K_z ,
$Ar^+(2) = T^+(3) \square SO(2)$ generated by P_t, P_x, P_z, J_y .

b) L(2,1) generated by J_y, K_x, K_z .

The subalgebras of L(2,1) are already known. According to Theorem III.5, the symmorphic subalgebras of $T^+(3) \square S(1)$ are :

a) $T^+(3) \square A(1)$ generated by P_t, P_x, P_z, A_x ,
$Ar^-(1,1) = T^+(3) \square L(1,1)$ generated by P_t, P_x, P_z, K_z .

b) $T^\circ(2) \square S(1)$ generated by $P_t - P_z, P_x, A_x, K_z$.

It is clear that we have obtained in this way all symmorphic four-dimensional subalgebras. It is also easy to prove that there is no nonsymmorphic subalgebra of dimension four.

Now, Theorem III.5 (parts a and b) together with Table IV.1 and IV.4 permit us to obtain the p.o.s. of symmorphic subalgebras of E(2,1). This p.o.s. appears to be a sub of $\mathcal{N}(P)$. The only difficulty concerns the subalgebras of nonsymmorphic algebras. We will examine this point later on.

Subalgebras of dimension three

The symmorphic ones are

L(2,1) generated by J_y, K_x, K_z .

$T^\circ(1) \square S(1)$ generated by $P_t - P_z, A_x, K_z$ (isomorphic to Th(2)) .

$T^\bullet(2) \square L(1,1)$ generated by $P_t - P_z, P_x, K_z$ (endowed with a direct product structure) .

$E(1,1) = T^+(2) \square L(1,1)$ generated by P_t, P_z, K_z .

$T^\bullet(2) \square A(1)$ generated by $P_t - P_z, P_x, A_x$ (isomorphic to Weyl algebra W(1)) .

$T^+(3)$ generated by P_t, P_x, P_z .

$E(2) = T^-(2) \square SO(2)$ generated by P_z, P_x, J_y .

Part c of Theorem III.5 can be used together with Theorems III.2 , III.3 , III.4 and Table IV.4 in order to get the nonsymmorphic subalgebras of dimension three. They are :

i) $Jor_\sigma(2) = [T^\bullet(1) \times A(1)] \square V_\sigma$ (20) generated by $P_t - P_z, A_x$,
$V_\sigma = \sigma K_z + P_x$, $(\sigma \neq 0)$.

(20) We can define the Jordan Lie algebra Jor(n) as follows. It is the semi-direct product $T(n) \square T(1)$ where T(n) is a n-dimensional abelian Lie algebra and such that the T(1) generator action on T(n) can be reduced to the Jordan matrix form:

$$\begin{pmatrix} 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$$

Its associate symmorphic algebras are $T^{\bullet}(1) \square S(1)$ and $T^{\bullet}(2) \square A(1)$. Note that $Jor_{\sigma}(2)$ and $Jor_{-\sigma}(2)$ are conjugate under a rotation of angle π around the z-axis.

ii) $T^{+}(2) \square V_{\sigma}$ generated by $P_t, P_z, V_{\sigma} = \sigma K_z + P_x$ $(\sigma \neq 0)$. Its associate symmorphic algebras are $T^{+}(2) \square L(1,1)$ and $T^{+}(3)$. A rotation of angle π around the z-axis change σ into $-\sigma$.

iii) $T^{\bullet}(2) \square A^{-}_{\varepsilon}(1)$ generated by $P_t - P_z, P_x$, $A^{-}_{x,\varepsilon} = A_x + \varepsilon(P_t + P_z)$, $(\varepsilon = \pm 1)$. Its associate symmorphic algebras are $T^{\bullet}(2) \square A(1)$ and $T^{+}(3)$. This algebra is isomorphic to $T^{\bullet}(2) \square A(1)$. Note that $T^{\bullet}(2) \square A^{-}(1)$ and $T^{\bullet}(2) \square A^{-}_{-\varepsilon}(1)$ are conjugate in $\hat{E}(2,1)$.

iv) $E^{-}_{\omega}(2) = T^{\bullet}(2) \square H^{-}_{\omega}$, isomorphic to $E(2)$, generated by $P_x, P_z, \omega J_y + P_t$ $(\omega \neq 0)$. Its associate symmorphic algebras are $E(2)$ and $T^{+}(3)$. Note that $E^{-}_{\omega}(2)$ and $E^{-}_{-\omega}(2)$ are conjugate in $\hat{E}(2,1)$ but not in $E(2,1)$.

In order to find the subalgebras of all these nonsymmorphic algebras, we have chosen a direct method.

Let us consider, for instance the case of $Jor_{\sigma}(2)$. The Lie brackets of this algebra are :

$$[P_t - P_z, V_{\sigma}] = -\sigma(P_t - P_z)$$
$$[P_t - P_z, A_x] = 0$$
$$[A_x, V_{\sigma}] = -\sigma A_x - (P_t - P_z) .$$

An element of this algebra reads

$$\alpha(P_t - P_z) + \beta A_x + \gamma V_{\sigma} .$$

Its invariants are $\theta = 0$, $\Delta = -\gamma^2\sigma^2$. According to Tables II.2, II.3, II.4 , we have the following classification of subalgebras of dimension one.

i) $\gamma = 0$ $(\Delta = 0)$ a) $\beta = 0$ pure translation $T^{\bullet}(1)$;
b) $\beta \neq 0$ $F^{*}.x = 0$ $A(1)$.

ii) $\gamma \neq 0$ $(U/\Delta^2 = -\frac{1}{\sigma^2})$ V_{σ} .

Now, by solving the following eigenvector problem

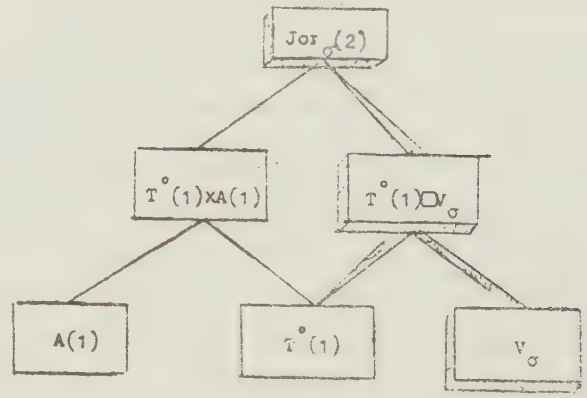
$$[\alpha(P_t - P_z) + \beta A_x + \gamma V_{\sigma} , X] = \lambda X .$$

We find :

i) $\lambda = 0$ $(\Rightarrow \gamma = 0)$ subalgebra generated by $P_t - P_z, A_x$.

ii) $\lambda \neq 0$ $(\Rightarrow \gamma \neq 0)$, $X = P_t - P_z$. There is only <u>one</u> subalgebra in the Poincaré Lie algebra generated by an element of type V_{σ} and a light-like translation. It is generated by $P_t - P_z$ and V_{σ} and isomorphic to $S(1)$.

The results are the following ones



Such a method permits us to complete $\mathcal{N}(\hat{E}(2,1))$.

It is given in Table V.2 . A star in a box denotes a nonsymmorphic algebra. Whenever a parameter σ or ω occurs, it labels an infinite set of nonconjugate subalgebras. The conditions $\sigma > 0$, $\omega > 0$ must be replaced by $\sigma \neq 0$, $\omega \neq 0$ if we are interested in conjugation under $E(2,1)$ transformations instead of Poincaré (or, equivalently, $\hat{E}(2,1)$) transformations. A similar property occurs in the cases of algebras having a subalgebra of type $A^{-}(1)$. Each box containing such an algebra splits into two boxes when we restrict ourselves to conjugation in the connected group $E(2,1)$.

For completeness, we now give the list of all subalgebras of dimension two contained in $E(2,1)$.

Subalgebras of dimension two .

The symmorphic ones are

$S(1)$	generated by	K_z, A_x ;
$T^-(1) \times L(1,1)$	generated by	P_x, K_z ;
$T^\circ(1) \square L(1,1)$	generated by	$P_t - P_z, K_z$;
$T^\circ(1) \times A(1)$	generated by	$P_t - P_z, A_x$;
$T^\circ(2)$	generated by	$P_t - P_z, P_x$;
$T^+(2)$	generated by	P_t, P_x ;
$T^-(2)$	generated by	P_x, P_z ;
$T^+(1) \times SO(2)$	generated by	P_t, J_y .

All are Abelian except $S(1)$ and $T^\circ(1) \square L(1,1)$ which are isomorphic.

The nonsymmorphic subalgebras are .

i) $T^\circ(1) \square V_\sigma$, $(\sigma \neq 0)$, isomorphic to $S(1)$. Its associate algebras are $T^\circ(1) \times L(1,1)$ and $T^\circ(2)$. The change of σ into $-\sigma$ is realized by a conjugation in $\hat{E}(2,1)$.

i) $T^\circ(1) \times A^-(1)$. Its associate algebras are $T^\circ(1) \times A(1)$ and $T^-(1)$.

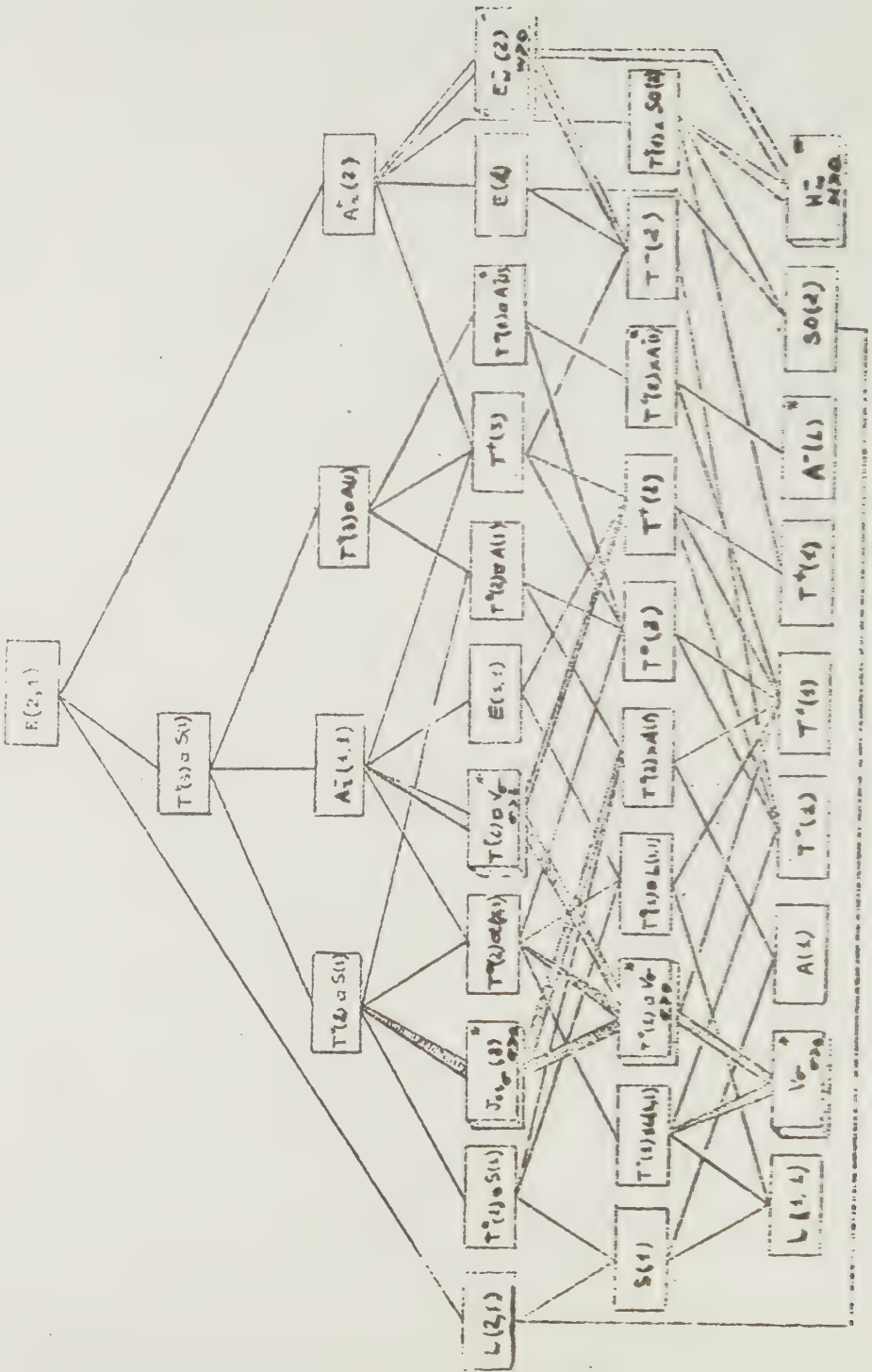


TABLE V.3 - CONNECTED SUBGROUPS OF $\hat{E}(2,1)$

A star means two things : i) nonsymmorphic character of the algebra
ii) splitting into two boxes when the lattice is the one of $E(2,1)$.

VI - SONS AND GRANDSONS OF THE POINCARE GROUP

The present section is devoted to the search of maximal connected subgroups of P (sons) and their maximal connected subgroups (grandsons of P) .

The first part of this program is relatively easy, due to the simplicity of Theorems III.6 and III.7 . According to these theorems, all sons are symmorphic. They are, apart the Lorentz group, all semi-direct products of T(4) by a maximal subgroup of the Lorentz group. Therefore, the sons of P are four in number

L(3,1) , the Lorentz group generated by $J_x, J_y, J_z, K_x, K_y, K_z$

IS(2) , the inhomogeneous S(2) group, generated by

$$P_t, P_x, P_y, P_z, J_z, K_z, A_x, A_y$$

$Ar^+(3) = T^+(1) \times E(3)$, the Aristotle group, generated by

$$P_t, P_x, P_y, P_z, J_x, J_y, J_z$$

$Ar^-(2,1) = T^-(1) \times E(2,1)$, generated by $P_t, P_x, P_y, P_z, J_z, K_x, K_y$.

A - MAXIMAL CONNECTED SUBGROUPS OF IS(2)

It is interesting to mention here that IS(2) contains all solvable connected subgroups of P . This is a direct consequence of the Borel-Morozov theorem for semi-simple complex Lie algebras[21].

(21) Let $\mathcal{P} = \mathcal{T} \square \mathcal{L}$ be an inhomogeneous Lie algebra with $\mathcal{L}$ isomorphic to a complex semi-simple Lie algebra. Let $\mathcal{B}$ the Borel subalgebra of $\mathcal{L}$ It is clear that $\mathcal{T} \square \mathcal{B}$ is solvable. If $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$ is a solvable subalgebra of $\mathcal{P}$, $\mathcal{L}_A + \mathcal{L}'_A$ is also solvable and, accordingly to the Borel-Morozov theorem, is conjugate to a subalgebra of $\mathcal{B}$ (conjugation in P is equivalent to conjugation in L) . Consequently $\mathcal{A}$ is a subalgebra of $\mathcal{T} \square \mathcal{B}$.

S(2) has two invariant subspaces in T(4) , namely $T^o(1)$ and $T^o(3)$ generated by $P_o - P_z$ and by $P_o - P_z$, P_x , P_y , respectively. According to Theorem III.5 and Table IV.2 , the maximal symmorphic subalgebras of IS(2) are

a) $T(4) \square {}^L E(2)$ generated by $P_t, P_x, P_y, P_z, J_z, A_x, A_y$

$T(4) \square S_f(2)$ generated by $P_t, P_x, P_y, P_z, J_f, A_x, A_y$

$T(4) \square Th(2)$ generated by $P_t, P_x, P_y, P_z, K_z, A_x, A_y$

$T(4) \square [SO(2) \times L(1,1)]$ generated by $P_t, P_x, P_y, P_z, J_z, K_z$

b) $T^o(3) \square S(2)$ generated by $P_t - P_z, P_x, P_y, J_z, K_z, A_x, A_y$.

We can prove that IS(2) has no nonsymmorphic maximal subalgebras in the following way. From remark which follows Theorem III.5 , the $\mathcal{L}$-associated algebra of such an algebra would be $T^o(3) \square S(2)$ which is seven-dimensional. Therefore, its t-associated algebra cannot be $T(4) \square [SO(2) \times L(1,1)]$ which is six-dimensional and $\mathcal{L}_A$ would be necessarily one of the following algebras ${}^L E(2)$, $S_f(2)$, $Th(2)$. Whatever the choice is, we have $[\mathcal{L}_A, \overline{\mathcal{L}'_A}] = 0$ and according to Theorem III.2 , $\overline{\mathcal{T}'_A}$ must commute with $\mathcal{L}_A$. If we refer now to Table IV.4 , it is easy to see that there is no solution.

B - MAXIMAL CONNECTED SUBGROUPS OF $Ar^+(3) = T^+(1) \times E(3)$

There is an easy problem since this group has a direct product structure with $\mathcal{T}$ of dimension one. We can then apply Theorem III.9 . We get, with the aid of Table V.1 , the symmorphic maximal subalgebras

a) $T^+(1) \times Ar^-(2) = T^+(4) \square SO(2)$ generated by P_t, P_x, P_y, P_z, J_z

$T^+(1) \times SO(3)$ generated by P_t, J_x, J_y, J_z

b) $E(3)$ generated by $P_x, P_y, P_z, J_x, J_y, J_z$.

Now, the derived Lie algebra of $Ar^+(3)$ is $E(3)$. Any nonsymmorphic algebra would be of the form $\mathcal{L}_A + \mathcal{M}_A$ with $\mathcal{L}_A$ of dimension 5

but we know that $E(3)$ has no subalgebra of dimension 5 . Therefore, there is no maximal nonsymmorphic subalgebra in $Ar^+(3)$.

- MAXIMAL CONNECTED SUBGROUPS OF $Ar^-(2,1) = T^-(1) \times E(2,1)$

The situation is quite similar to that of $Ar^+(3)$. The maximal subalgebras are all symmorphic.

a) $T^-(1)\times[T^+(3)\supset S(1)] = T(4)\supset S(1)$ generated by $P_t, P_x, P_y, P_z, K_z, A_x$

[illegible] generated by P_t, P_x, P_y, P_z, K_z

$T^-(1) \times L(2,1)$ generated by P_z, J_z, K_x, A_y

[illegible] generated by $P_t, P_x, P_y, P_z, K_x, A_y$

All the above results are put together in Table VI.1 . Let us add the following remarks :

) all sons and grandsons of P are symmorphic or homogeneous.

) any [illegible] of dimension larger than six is either a son or a grandson [illegible] P .

) a subgroup of a given group cannot be a son and a grandson but [illegible] simultaneous different kinships can occur. For instance SO(3) [illegible] both a grandson and a grandgrandson of P since we have the two chains:

[illegible] $\supset E(3) \supset SO(3)$

$P \supset L(3,1) \supset SO(3)$

[illegible] symbol $\supset$ denotes a "maximal inclusion". Every such inclusion which does not correspond to a son or grandson kinship is represented by a dashed line. An arrow means that the corresponding subset is already known.

We have thus obtained the top of $\mathcal{M}(\mathcal{P})$. If we take into account of the preceding results, namely the bottom of $\mathcal{M}(\mathcal{P})$ and the p.o.s. of $L(3,1)$, $T(4)$, $E(3)$ and $E(2,1)$ we have fulfilled 70 "boxes"

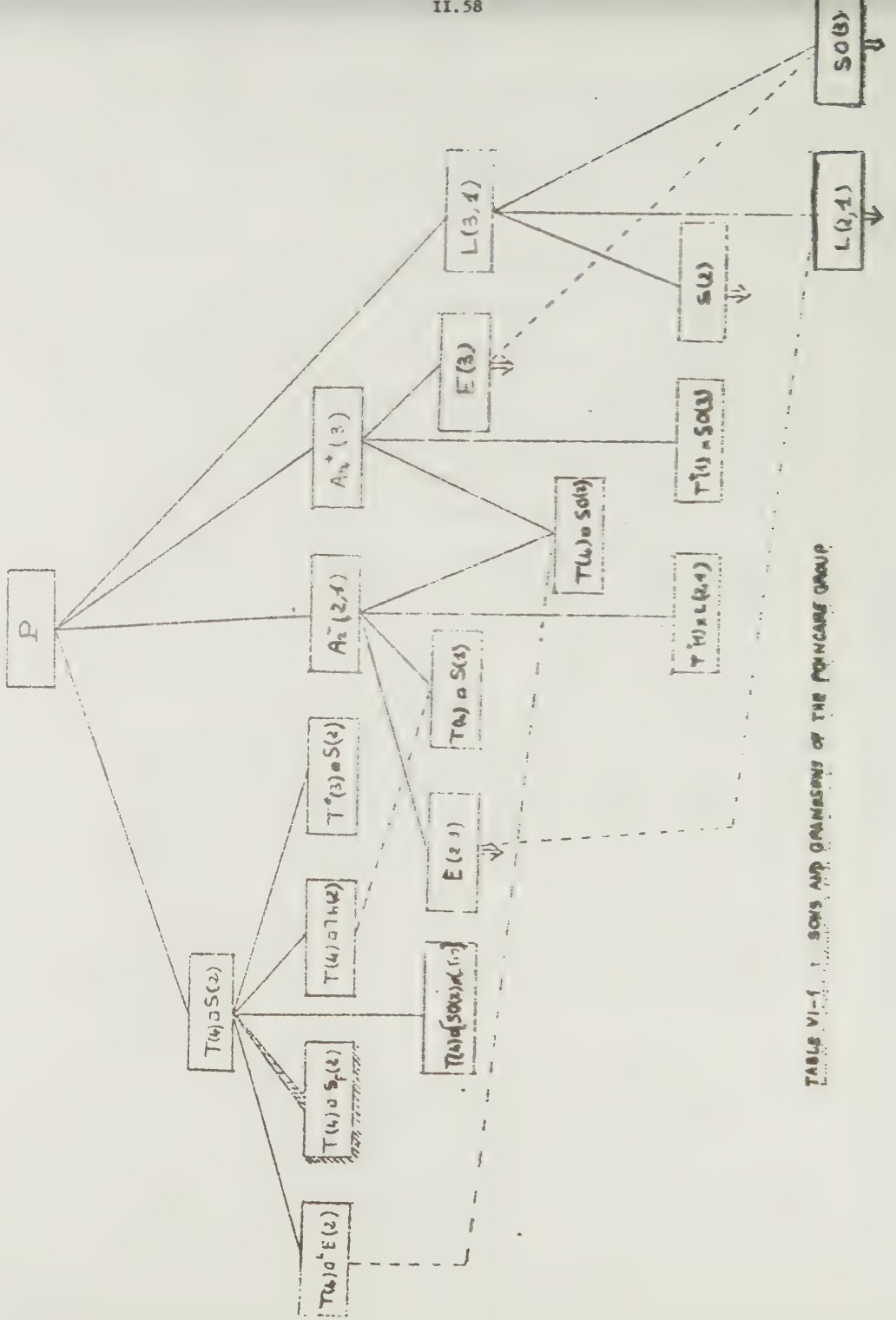


TABLE VI-1 : SONS AND GRANDSONS OF THE POINCARÉ GROUP

In order to give an idea of how "large" is $\mathcal{N}(\mathcal{P})$ we must say that there are around 140 boxes on the whole. It will be completed in a forthcoming paper.

ACKNOWLEDGMENTS

The authors are grateful to L. Michel, C. Ridoux and L. Sirugue for some discussions on the subject, and A. Grossmann for a serious critical reading of the manuscript.

APPENDIX A

ON AUTOMORPHISMS OF INHOMOGENEOUS LIE ALGEBRAS

Lemma

Let $\mathcal{P} = \mathcal{T} \square \mathcal{L}$ the Lie algebra of an inhomogeneous Lie group $P = T \square L$. If $\mathcal{L}$ is semi-simple, every automorphism σ of $\mathcal{P}$ is such that $\sigma(\mathcal{T}) = \mathcal{T}$.

Proof : Define $\sigma(\mathcal{T}) = \mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{U}_A$ with $\mathcal{L}_A + \mathcal{U}_A \neq 0$. Because $\mathcal{A}$ is Abelian, $\mathcal{L}_A + \mathcal{L}'_A$ is also Abelian. Moreover, $\sigma(\mathcal{T})$ must be an ideal of $\mathcal{P}$:

$$[\mathcal{P},\mathcal{A}] \subseteq \mathcal{A} \Rightarrow [\mathcal{L},\mathcal{A}] \subseteq \mathcal{A} \Rightarrow [\mathcal{L},\mathcal{L}_A+\mathcal{L}'_A] \subseteq \mathcal{L}_A+\mathcal{L}'_A$$

and $\mathcal{L}_A+\mathcal{L}'_A$ would be a proper Abelian ideal of $\mathcal{L}$, in contradiction with our assumption (i.e. $\mathcal{L}$ to be semi-simple).

We are interested in finding the connected component of the group of automorphisms of $\mathcal{P}$. According to the lemma, $\sigma(\mathcal{P}) = \mathcal{T} + \sigma(\mathcal{L})$, where $\sigma(\mathcal{L})$ is a complementary subspace, i.e. $\sigma(\mathcal{L}) \cap \mathcal{T} = 0$; therefore, $\sigma(\mathcal{L}) = \mathcal{L}_B + \mathcal{U}_B$ with $\mathcal{L}_B + \mathcal{L}'_B = \mathcal{L}$. Now, Theorem III.4 tells us that there exists an inner automorphism σ_o mapping $\mathcal{L}$ onto $\sigma(\mathcal{L})$. Since any connected automorphism of a semi-simple Lie algebra is an inner automorphism (see reference [42]), the automorphism $\tau = \sigma_o^{-1}\sigma$ is equal to the identity operator when restricted to $\mathcal{L}$.

$$\tau(\mathcal{P}) = \mathcal{T} + \mathcal{L}$$

and

$$\forall \ell \in \mathcal{L}, \; t \in \mathcal{T} \; ; \quad \tau[\ell,t] = [\tau(\ell),\tau(t)] = [\ell,\tau(t)] .$$

Since $\mathcal{L}$ is semi-simple, its adjoint action on $\mathcal{T}$ is completely reducible, i.e. $\mathcal{T} = \bigoplus_i \mathcal{T}_i$. Denote by $\mathrm{ad}_i(\ell)$ the action

of the element $\mathcal{L}$ on $\mathcal{T}_i$; we get:

$$\tau \, ad_{\mathcal{L}}(\mathcal{L}) = ad(\cdot)\tau \quad , \quad \forall \mathcal{L} \in \mathcal{L} \ .$$

From Schur's lemma, τ is a multiple of the identity when restricted to $\mathcal{T}_i$.

Theorem A.4 (22)

The Lie algebra of the group of automorphisms of the inhomogeneous Lie algebra $\mathcal{P} = \mathcal{T} \square \mathcal{L}$, where $\mathcal{L}$ is semi-simple, is isomorphic to $(\mathcal{T} \square \mathcal{L}) \square S$ where S is [illegible] automorphisms or dilatations in each invariant subspace $\mathcal{T}_i$ under $\mathcal{L}$, provided that the decomposition of $\mathcal{T}$ into $\mathcal{T}_i$ is unique.

For obtaining the present proof of this theorem, the authors have been helped by another proof given by S. Racanati (private communication).

APPENDIX B

THEOREMS ON SEMI-SIMPLE LIE ALGEBRAS

Theorem B.1

Let $\mathcal{A}$ be a real semi-simple compact Lie algebra of dimension n ; it has no subalgebra of dimension $n-1$.

Theorem B.2

A real simple Lie algebra of dimension n has no $(n-1)$-dimensional subalgebra, except $SO(2,1)$ which has a two-dimensional subalgebra.

Theorem B.3

If $\mathcal{P} = \mathcal{T} \square \mathcal{L}$ is an inhomogeneous real Lie algebra of dimension n with $\mathcal{L}$ simple and acting irreducibly on $\mathcal{T}$, the condition for $\mathcal{P}$ to have an $(n-1)$-dimensional subalgebra is $\mathcal{L}$ to be isomorphic to $SO(2,1)$ and/or $\dim \mathcal{T} = 1$.

The reader will easily check these theorems in the cases of the Lorentz Lie algebra which has no 5-dimensional subalgebra (see Table IV.2) and of the Poincaré Lie algebra which has no 9-dimensional subalgebra (see Table VI.1) .

We intend to give in the present Appendix a complete proof of these theorems. In fact, Theorem B.3 is an immediate consequence of Theorems B.1 and B.2 and Theorem III.6 .

Let us denote by (x,y) the Killing bilinear product in a Lie algebra $\mathcal{A}$. The invariance property of the Killing form reads

(B.1) $$\forall x,y,z \in \mathcal{A} \quad ([x,y],z) + (y,[x,z]) = 0 \ .$$

We denote by $\mathfrak{B}^{\perp}$ the subspace which is orthogonal to a given subspace $\mathfrak{B}$ of $\mathcal{A}$

(B.2) $$\mathfrak{B}^{\perp} = \{\beta \in \mathcal{A} \mid \forall b \in \mathfrak{B},\ (\beta,b) = 0\}$$

This definition can be written in short

(B.3) $$(\mathfrak{B},\mathfrak{B}^{\perp}) = 0 .$$

Let us first recall the following well-known lemmas for a real Lie algebra $\mathcal{A}$

Lemma 1

If $\mathfrak{B}$ and $\mathfrak{F}$ are subspaces of $\mathcal{A}$

$$\mathfrak{B} \subseteq \mathfrak{F} \Rightarrow \mathfrak{F}^{\perp} \subseteq \mathfrak{B}^{\perp}$$

Lemma 2

If $\mathfrak{B}$ and $\mathfrak{F}$ are subspaces of $\mathcal{A}$

$$(\mathfrak{B} + \mathfrak{F})^{\perp} = \mathfrak{B}^{\perp} \cap \mathfrak{F}^{\perp}$$

Lemma 3

If $\mathfrak{B}$ and $\mathfrak{F}$ are ideals of $\mathcal{A}$, $\mathfrak{B} \cap \mathfrak{F}$ is an ideal of $\mathcal{A}$.

Lemma 4

If $\mathfrak{B}$ is an ideal of $\mathcal{A}$, $\mathfrak{B}^{\perp}$ is also an ideal of $\mathcal{A}$.

This property follows form Eq (B.1) with $x \in \mathcal{A}$, $y \in \mathfrak{B}$, $z \in \mathfrak{B}^{\perp}$.

Lemma 5

If $\mathfrak{B}$ is a subalgebra of $\mathcal{A}$, $\mathfrak{B}^{\perp}$ and $\mathfrak{B} \cap \mathfrak{B}^{\perp}$ are ideals of $\mathfrak{B}$.

This is easily proved in using Eq(B.1) with $x,z \in \mathfrak{B}$ and $y \in \mathfrak{B}^{\perp}$ and Lemma 3 .

Lemma 6

If $\mathcal{A}$ is semi-simple and of dimension n

$$\mathfrak{B} \subseteq \mathcal{A} \Rightarrow (\mathfrak{B}^{\perp})^{\perp} = \mathfrak{B} \ , \ \dim \mathfrak{B} + \dim \mathfrak{B}^{\perp} = n .$$

This is a direct consequence of the nondegenerate character of the Killing form for semi-simple Lie algebras.

Lemma 7

If $\mathcal{A}$ is semi-simple, any ideal of $\mathfrak{B}$ is semi-simple.

For the proof, see for instance Melgasson [49], p. 121 .

Lemma 8

If $\mathcal{A}$ is compact, for any subspace $\mathfrak{B}$, we have $\mathfrak{B} \cap \mathfrak{B}^{\perp} = 0$

This property is a direct consequence of the definiteness of the Killing form.

We can now prove Theorems B.1 and B.2 . Let us consider a real semi-simple Lie algebra $\mathcal{A}$ of dimension n and suppose there exists an (n-1)-dimensional subalgebra $\mathfrak{B}$.

We immediately note that $n \geq 3$ (that is no lower dimensional semi-simple algebra). Let us consider the two cases :

Case 1 : $\mathfrak{B} + \mathfrak{B}^{\perp} = \mathcal{A}$

From Lemma 6 , $\dim \mathfrak{B}^{\perp} = 1$ and from Lemma 5 , $\mathfrak{B}^{\perp}$ is an ideal of $\mathfrak{B}$ Since $[\mathfrak{B}^{\perp}\ \mathfrak{B}^{\perp}] = 0$, it is also an Abelian ideal of $\mathcal{A}$ in contradiction with the semi-simple character of $\mathcal{A}$. We have then proved Theorem B.1 since, according to Lemma 8 , if $\mathcal{A}$ is compact, we necessarily have $\mathfrak{B} + \mathfrak{B}^{\perp} = \mathcal{A}$.

Case 2 : $\mathfrak{B} + \mathfrak{B}^{\perp} \neq \mathcal{A}$

This implies $\dim(\mathfrak{B}+\mathfrak{B}^{\perp}) < n$ and $\mathfrak{B}+\mathfrak{B}^{\perp} = \mathfrak{B}$. Lemma 6 gives $\dim \mathfrak{B}^{\perp} = 1$ and Lemma 5 tells us that $\mathfrak{B}^{\perp}$ is an ideal of $\mathfrak{B}$. The quotient algebra $\mathfrak{B}/\mathfrak{B}^{\perp}$ is isomorphic to some subalgebra $\mathcal{X}$ of $\mathfrak{B}$ with dimension n-2 (we remind the reader that $n-2 \geq 1$) and $\mathcal{X} + \mathfrak{B}^{\perp} = \mathfrak{B}$. From Lemma 5 , $\mathcal{X}^{\perp}$ is an ideal of $\mathcal{X}$. Now, $\mathcal{X} \subset \mathfrak{B}$ implies $\mathfrak{B}^{\perp} \subset \mathcal{X}^{\perp}$ (Lemma 1) and $[\mathcal{X},\mathfrak{B}^{\perp}] \subset \mathfrak{B}^{\perp}$. Lemma 2 gives the property

$$\mathcal{X} + \mathcal{B}^{\perp} = \mathcal{B} \quad \Rightarrow \quad \mathcal{X}^{\perp} \cap \mathcal{B} = \mathcal{B}^{\perp}$$

and, consequently, since $\mathcal{X}$ does not contain $\mathcal{B}^{\perp}$

(B.4) $$\mathcal{X} \cap \mathcal{X}^{\perp} = 0$$

and

(B.5) $$\mathcal{X} + \mathcal{X}^{\perp} = \mathcal{A} \quad .$$

There exists in $\mathcal{X}^{\perp}$ an isotropic vector which is not contained in $\mathcal{B}^{\perp}$

(Proof : let $b_o \in \mathcal{B}^{\perp}$ $(b_o \neq 0)$ and $x_o \in \mathcal{X}^{\perp}$, $x_o \notin \mathcal{B}^{\perp}$. If $(x_o,x_o) \neq 0$, we can always find λ such that $(x_o + \lambda b_o, x_o + \lambda b_o) = (x_o,x_o) + 2\lambda(x_o,b_o) = 0$ because $(x_o,b_o) = 0$ would imply the orthogonality of $\mathcal{B}^{\perp}$ to all elements of $\mathcal{A}$, in contradiction with the semi-simple character of $\mathcal{A}$) . Let us denote by $\mathcal{C}^{\perp}$ this isotropic direction. The relation $\mathcal{C}^{\perp} \subset \mathcal{X}^{\perp}$ implies $\mathcal{X} \subset \mathcal{C}$. Since $\mathcal{C} \supset \mathcal{C}^{\perp}$ and is (n-1)-dimensional, we get

(B.6) $$\mathcal{C} = \mathcal{X} + \mathcal{C}^{\perp}$$

Now, $\mathcal{C}^{\perp} \subset \mathcal{X}^{\perp}$ implies (remember that $\mathcal{X}^{\perp}$ is an ideal of $\mathcal{X}$),

(B.7) $$[\mathcal{X},\mathcal{C}^{\perp}] \subset [\mathcal{X},\mathcal{X}^{\perp}] \subseteq \mathcal{X}^{\perp}$$

Eq(B.1) with $x,z \in \mathcal{C}^{\perp}$, $y \in \mathcal{X}$ shows us that

(B.8) $$[\mathcal{X},\mathcal{C}^{\perp}] \subset \mathcal{C}$$

From (B.7) and (B.8), we obtain, with the aid of Lemma 2 and Eq(B.6)

(B.9) $$[\mathcal{X},\mathcal{X}^{\perp} \cap \mathcal{C}^{\perp}] \subseteq \mathcal{X}^{\perp} \cap \mathcal{C} = (\mathcal{X} + \mathcal{C}^{\perp})^{\perp} = \mathcal{C}^{\perp}$$

a relation which proves that $\mathcal{C}^{\perp}$ is an ideal of $\mathcal{C}$. Therefore, there is a complete symmetry between $\mathcal{B}$ and $\mathcal{C}$, namely

(B.10) $$\begin{cases} [\mathcal{B}, \mathcal{B}^{\perp}] \subseteq \mathcal{B}^{\perp} \quad , \quad \mathcal{B}^{\perp} \subset \mathcal{B} \\ \\ [\mathcal{C}, \mathcal{C}^{\perp}] \subseteq \mathcal{C}^{\perp} \quad , \quad \mathcal{C}^{\perp} \subset \mathcal{C} \end{cases} \qquad \mathcal{B} \cap \mathcal{C} = \mathcal{X}$$

Eq(B.1) can be applied once more with $x,z \in \mathcal{B}^{\perp}$, $y \in \mathcal{C}^{\perp}$ (and also with $x,y \in \mathcal{C}^{\perp}$, $z \in \mathcal{B}^{\perp}$) . We get

(B.11) $$\begin{cases} [\mathcal{B}^{\perp}, \mathcal{C}^{\perp}] \subseteq \mathcal{B} \\ \\ [\mathcal{B}^{\perp}, \mathcal{C}^{\perp}] \subseteq \mathcal{C} \end{cases}$$

Therefore

(B.12) $$[\mathcal{B}^{\perp},\mathcal{C}^{\perp}] \subseteq \mathcal{B} \cap \mathcal{C} = \mathcal{X}$$

The bracket $[\mathcal{B}^{\perp},\mathcal{C}^{\perp}]$ cannot be zero ; if it was, $\mathcal{X}^{\perp}$ would be an Abelian ideal of $\mathcal{A}$ which is known to be impossible. Let $\mathcal{J}$ the one-dimensional subalgebra defined by

(B.13) $$[\mathcal{B}^{\perp},\mathcal{C}^{\perp}] = \mathcal{J}$$

We have $\mathcal{J} \subseteq \mathcal{X}$. It is now easy to prove that $\mathcal{B}^{\perp} \oplus \mathcal{C}^{\perp} + \mathcal{J}$ is an ideal of $\mathcal{A}$ by the use of Jacobi's identity. From Lemma 7 , it follows that the 3-dimensional algebra generated by $\mathcal{B}^{\perp}$, $\mathcal{C}^{\perp}$ and $\mathcal{J}$ is semi-simple. Since

$$[\mathcal{J},\mathcal{B}^{\perp}] \subseteq \mathcal{B}$$

$$[\mathcal{J},\mathcal{C}^{\perp}] \subseteq \mathcal{C}$$

it is the SO(2,1) Lie algebra. Theorem B.2 readily follows.

APPENDIX C

DEFINITION OF THE MOTION OF A PARTICLE IN A HOMOGENEOUS EXTERNAL ELECTROMAGNETIC FIELD

Let θ be a dimensionless "time" evolution parameter, which is proportional to the proper time in the case of tardyons[23]. A world line $x(\theta)$ is defined by $x(0)$, $\frac{dx}{d\theta}(0) = X\,x(0)$ where X is a dismensionless operator. By integration, we have

$$x(\theta) = e^{\theta X}\, x(0) .$$

A condition for $x(0)$ to be a trajectory of a charged particle in a homogeneous field F is $\frac{d^2x}{d\theta^2}$ to be proportional to $F \cdot \frac{dx}{d\theta}$, i.e. $X^2\, x(0)$ to be parallel to $F \cdot X\, x(0)$.

It is clear that the general element $X = \frac{e}{2} F^{\mu\nu} M_{\mu\nu} + a_\mu P^\mu$ of the Lie algebra of the Poincaré group which acts on x^λ in the following way :

$$X\, x^\lambda = e\, F^{\lambda\mu} x_\mu + a^\lambda$$

satisfies such a condition. Therefore, it is possible to define the world line of a charged particle of any kind (tardyon, tachyon or luxyon*) in a homogeneous external field $F_{\mu\nu}$, as an orbit on Minkowski space under a one-dimensional connected subgroup of P. It can be shown, conversely, that each such orbit represents a worldline of some kind of particle in a given field and with given initial conditions.

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(23) A recent terminology for particles of speed smaller than c. Tachyons have speed larger than c and luxyons speed c.

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CHAPITRE 2 : CONNECTED SUBGROUPS OF THE POINCARE GROUP II

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ABSTRACT : The determination of connected subgroups of the Poincaré group P , begun in a previous paper, is completed. Several tables are given where the subalgebras of dimension two to six are classified up to a conjugation and up to an isomorphism. In our method, a great use is made of tensor calculus, whose main results are recalled or stated in the beginning of the paper.

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C O N T E N T S

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(Continued)

VII - INTRODUCTION

VIII - CONNECTED SUBGROUPS OF DIMENSION TWO

IX - CONNECTED SUBGROUPS OF DIMENSION THREE

X - CONNECTED SUBGROUPS OF DIMENSION FOUR, FIVE AND SIX

XI SONS AND GRANDSONS OF THE SIX-DIMENSIONAL CONNECTED SUBGROUPS OF THE POINCARE GROUP

INTRODUCTION

-:-:-:-:-:-:-

In a previous paper [1], informations have been given on some parts of $\mathcal{N}(\mathcal{P})$, the p.o.s. of the subalgebras up to a conjugation of the Lie algebra $\mathcal{P}$ of the Poincaré group P. More precisely, the "members" of $\mathcal{N}(\mathcal{P})$ studied in that work were :

- the one-dimensional subalgebras of $\mathcal{P}$, that is the bottom of $\mathcal{N}(\mathcal{P})$.
- the maximal subalgebras of $\mathcal{P}$, that is the top of $\mathcal{N}(\mathcal{P})$.
- some sublattices which are lattices of some subalgebras, namely the Lorentz algebra $\mathcal{L}$ and the algebra of the Euclidean group $E(3)$.
- A sublattice which is "almost" the lattice of $E(2,1)$, the Poincaré group in a three-dimensional space time.
- The part of $\mathcal{N}(\mathcal{P})$ which only contains translations.

The aim of this paper is to complete the "body" of $\mathcal{N}(\mathcal{P})$.

i) In sections VIII and IX, the two- and three-dimensional subalgebras are obtained with the aid of the classification of Lie algebras up to an isomorphism [2].

ii) Section X is devoted to the four-, five- and six-dimensional algebras, which are built with the help of the results states in ref. [1].

iii) In section XI, we construct for each six-dimensional subalgebra a table giving its sons and grandsons [1].

Let us note, moreover, that for each dimension, we gather conjugation classes in isomorphism classes.

We complete this introduction by realling the main notationas used in ref. [1] and some properties of tensor algebra on Minkowsky space.

A) - TENSOR CALCULUS IN $\mathcal{P}$

As in Paper I ,

(VII.1) $$(x,F) = x^\mu P_\mu + \frac{1}{2} F^{\mu\nu} M_{\mu\nu}$$

denotes an element of the Poincaré Lie algebra where the P_μ's and $M_{\mu\nu}$'s are the usual generators of the Poincaré group (each couple (x,F) , built with a four-vector x and a tensor of components $F^{\mu\nu} = -F^{\nu\mu}$, corresponding to a single element of the Lie algebra).

With this notation, the Lie braket of two elements (x,F) and (x',F') of the Poincaré Lie algebra becomes

(VII.2) $$\left[(x,F),(x',F')\right] = \left(x.F' - x'.F\ ,\ F\wedge F'\right)$$

where

(VII.3) $$(x.F)^\mu = x_\nu F^{\nu\mu}$$

and the skew symmetric tensor of rank two (s.s.t.t.) $F\wedge F'$ is defined by

(VII.4) $$(F\wedge F')^{\mu\nu} = F^{\mu\rho} F'^{\ \nu}_{\rho} - F^{\nu\rho} F'^{\ \mu}_{\rho}$$

B) - PLANE TENSOR [3] [4]

Let us recall that every s.s.t.t. F on the Minkowsky vector space $\mathcal{M}$ can be put in the form

(VII.5) $$F = a\wedge b + c\wedge d$$

A tensor F is said to be plane if it can be written in the form

(VII.6) $$F = a\wedge b$$

We shall call F-plane the 2-plane spanned by the two 4-vectors a and b . There exists a mapping of plane tensors onto the set of 2-dimensional subspaces of $\mathcal{M}$.

Let us define, as in [1], the two invariant quantities

(VII.7) $$\Delta(F) = \frac{1}{2} F^{\mu\nu} F_{\mu\nu}$$

and

(VII.8) $$\phi(F) = \frac{1}{2} F^{*\mu\nu} F_{\mu\nu}$$

where

(VII.9) $$F^{*\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\rho\lambda} F_{\rho\lambda}$$

We can remark that $\phi(F) = 0$ if and only if F is plane. Moreover, we can classify the plane tensors according to the type of the F-plane, that is to the sign of $\Delta(F)$

$\Delta(F) > 0$ the F plane is space like ;

$\Delta(F) = 0$ the F plane is tangent ;

$\Delta(F) < 0$ the F plane is hyperbolic .

Let us now give some important properties satisfied by s.s.t.t.

Theorem VII.1

If F is plane , so is F^* .

Theorem VII.2

F being plane :

i) If F is space-like, F^* is hyperbolic ; and if F is hyperbolic, F^* is space-like. The F plane and the F^* plane are complementary.

ii) If F is tangent, so is F^* . The F plane and the F^* plane span a tangent hyperplane (3-dimensional subspace of $\mathcal{M}$).

Theorem VII.3

Any s.s.t.t. F can be written in the form

(VII.10) $$F = i\beta\, e_+\wedge e_- + \varepsilon\, e\wedge e'$$

where $e_\pm$ are complex four vectors such that $e_+^2 = e_-^2 = 0$, $e_+.e_- = -1$ and e,e' real four vectors such that $e^2 = e'^2 = 0$, $e.e' = 1$. Moreover : $e_+.e = e_+.e' = e_-.e = e_-.e' = 0$. The real scalars β and ε are given by

(VII.11) $$\beta = \sqrt{\frac{\Delta + \sqrt{\Delta^2+\phi^2}}{2}} \qquad \varepsilon = \sqrt{\frac{-\Delta + \sqrt{\Delta^2+\phi^2}}{2}}$$

and verify

$$\begin{cases} F.e = \ell e & F.e_+ = i\beta e_+ \\ F.e' = -\ell e' & F.e_- = -i\beta e_- . \end{cases}$$

Theorem VII.4 [5]

F being plane :

i) If F is space-like, it can be written as

(VII.12) $F = i\beta\, e_* \wedge e.$

with $\beta = \sqrt{\Delta}$

ii) If F is hyperbolic, it can be written as

(VII.13) $F = \ell\, e \wedge e'$

with $\ell = \sqrt{-\Delta}$

iii) If F is tangent, it can be written as

(VII.14) $F = a \wedge l$

where $l^2 = a.l = 0 \quad , \quad a^2 < 0.$

In each case, the vectors of F^* plane are the eigenvectors of F of eigenvalue 0 .

From this theorem, we can easily deduce the following results.

Theorem VII.5

If F is plane and x an arbitrary four vector :

i) The vector F.x lies in the F plane.

ii) There exists a vector y such that x - F.y lies :

- in the F^* plane if $\Delta'(F) \neq 0$
- in the complementary plane of the F plane if $\Delta(F) = 0$

C) - SOLUTIONS OF F.x = 0

In the following section, we will be interested in the vectors x solutions of the equation

(VII.15) $F^{\mu\nu}.x_\nu = 0$

where F is a given a.e.t.t.

Using the results of Appendix A of ref [5] , we can solve the problem. The solutions are summarized in Table (VII.1) .

Type of F		Type of the solution		Type of $F^*.x$
$\phi(F) \neq 0$		$x = 0$		$F^*.x = 0$
$\phi(F) = 0$, $F \neq 0$ (F and F^* are plane and orthogonal)	$\Delta(F) > 0$ F-plane is space-like	$x = 0$		$F^*.x = 0$
		$x \neq 0$, $x \in F^*$-plane (F^*-plane is hyperbolic)	$x^2 > 0$	$(F^*.x)^2 < 0$
			$x^2 = 0$	$(F^*.x)^2 = 0$ with $F^*.x \neq 0$
			$x^2 < 0$	$(F^*.x)^2 > 0$
	$\Delta(F) < 0$ F-plane is hyperbolic	$x = 0$		$F^*.x = 0$
		$x \neq 0$, $x \in F^*$-plane (F^*-plane is space-like)	$x^2 < 0$	$(F^*.x)^2 < 0$
	$\Delta(F) = 0$ F-plane is tangent	$x = 0$		$F^*.x = 0$
		$x \neq 0$, $x \in F^*$-plane (F^*-plane is tangent)	$x^2 = 0$	$F^*.x = 0$
			$x^2 < 0$	$(F^*.x)^2 = 0$ with $F^*.x \neq 0$
$F = 0$		x arbitrary		$F^*.x = 0$

Table VII.1 : Solutions of $F^*.x = 0$

D) - SOLUTIONS OF $F.x = \pm x$

We will also be interested in those vectors x which are solutions of the equation

(VI.16) $F.x = \pm x$

herein x an eigenvector of F whith eigenvalue $\pm i$. From Theorem VII.3, x is necessarily 0 or light like.

The eigenvalues are roots of the equation

(VII.17) $\det\left[F_{\mu\nu} - \lambda\, g_{\mu\nu}\right] = 0$

or

(VII.18) $\lambda^2 = -1 \pm \sqrt{\Delta - 1}$

By replacing λ by $\pm i$, one obtains the condition

(VII.19) $\Delta = 1$

It follows that Eq. (VII.18) is only possible if F satisfies one of the two following conditions :

i) $\Phi(F) \neq 0$

ii) $\Phi(F) = 0$ and $\Delta(F) = +1$.

2) - SOLUTIONS OF $G \wedge F = 0$ AND $G \wedge F = F$

Theorem VII.4 :

The product of two s.a.s.t.c. $G \wedge F \quad (F \neq 0)$ is zero if and only if G is a linear combination of F and F^*.

The proof is given in ref. [5]

Let us consider now the three following lemmas which will allow a covariant demonstration of Theorem VII.7

Lemma 1 :

(VII.20) $(A \wedge B).C = (B \wedge C).A = (C \wedge A).B$

Lemma 2 :

and

(VII.21) $\nabla_\mu B^{\alpha\beta}{}_\lambda = \partial_\mu A^{\alpha}{}_\lambda + \frac{1}{2}(A.B)\, g_{\alpha\lambda}$

Lemma 3 :

F being s.a.s.t.c.

i) $F.x = F^*.x = 0$ and $x \neq 0 \Rightarrow \Phi(F) = \Delta(F) = 0$ and $x^2 = 0$

ii) $\Phi(F) = \Delta(F) = 0 \Rightarrow \exists\, x \neq 0$,

x unique up to a factor, such that $x^2 = 0$ and $F.x = F^*.x = 0$.

Lemma 4 :

A and B being s.a.s.t.c.

(VII.22) $(A \wedge B)^* = A^* \wedge B = A \wedge B^*$.

Theorem VII.7 :

Given a nonzero s.a.s.t.c. F, the equation

(VII.23) $G \wedge F = F$

implies $\Delta(F) = \Phi(F) = 0$

$\Phi(G) = 0$; $\Delta(G) = -1$

The most general solution of Eq. (VII.23) reads

(VII.24) $G = G_0 + \alpha F + \beta F^*$

where G_0 is a particular solution and α and β are real.

The proof of this theorem is as follows.

By contraction of the two members of the Eq. (VII.23) successively by F and F^*, and using lemma 1 and Theorem VII.6, we deduce easily $\Delta(F) = \Phi(F) = 0$. By contraction of the same tensors by G and G^* respectively and using the same properties, we obtain :

(VII.25) $F.G = F.G^* = 0$

from which we can deduce :

(VII.26) $F^*.G^* = F^*.G = 0$

Let us now remark that if G is a solution of Eq. (VII.23), G is a solution of

(VII.27) $G \wedge (G \wedge F) = F$

Then using lemma 2 and Eqs. (VII.25) and (VII.26), we obtain :

(VII.28) $G \wedge (G \wedge F) = -\Delta(G)\, F + \Phi(G)\, F^*$

It follows, from Eq. (VII.27) that :

$$\Delta(G) = -1 \quad \text{and} \quad \phi(G) = 0$$

Let us now study the general solution of G for a given F. We can note that if G_0 is a solution of Eq. (VII.23), $G_0 + \alpha F + \beta F^*$, α and β real, is also a solution. This property is an immediate consequence of Theorem VII.6.

Lemma 3 insures the existence of a light like vector l $(l^2 = 0)$ unique up to a factor such that

(VII.29) $$F.l = F^*.l = 0$$

It follows that $(G \wedge F).l = 0$ from which we deduce

(VII.30) $$F.(G.l) = 0$$

We can use lemma 4 to show that if G is a solution of Eq. (VII.23) then G is also a solution of

(VII.31) $$G \wedge F^* = F^*$$

It follows that

(VII.32) $$F^*.(G.l) = 0$$

Eqs. (VII.30) and (VII.32) imply

(VII.33) $$G.l = \lambda l$$

and Table VII.1 shows that λ cannot be zero.

We can then write [5] F and G_0 on the following forms

(VII.34) $F = l \wedge a$ with $l.a = 0$, $a^2 < 0$

(VII.35) $G_0 = l \wedge l'$ with $l'^2 = 0$

Eq. (VII.23) shows that $l.l' = 1$.

VIII - CONNECTED SUBGROUPS OF DIMENSION TWO

It is well-known [2] that there are only two non isomorphic Lie algebras of dimension two. Their canonical forms are either :

(VIII.1) $$[A,B] = 0$$

(Abelian algebra, noted $(g_{2,1})^2$ in ref. [6] and $4g_1$ in ref. [7]).

or :

(VIII.2) $$[A,B] = B$$

(the algebra of $S(1)$, noted g_{2} in ref. [6] and [7]).

A) - ABELIAN ALGEBRAS

According to Theorem (VII.6), we can choose as generators of any Abelian algebra (x,F) and $(x',\lambda F^*)$ with the relation :

(VIII.3) $$F.x' = \lambda F^*.x$$

where λ can be given values 0 or 1.

Let us consider the three cases $F = 0$; $F \neq 0$, $\lambda = 0$; and $F \neq 0$, $\lambda = 1$.

1. Case $F = 0$

Translational algebras corresponding to $F = 0$, have already been studied in section IV, §A. We recall these algebras in the three first rows of Table VIII.1.

2. Case $F \neq 0$, $\lambda = 0$.

The generators are then (x,F) and $(x',0)$. The relation (VIII.3) becomes :

(VIII.4) $$F.x' = 0.$$

As a consequence of Table VII.1, we have $\phi(F) = 0$ and x' belongs to the F^* plane.

By an inner automorphism, we can transform the element (x,F) into one of the elements of the first column of Table II.5. (Let us note that (x,F) is neither

J_ξ nor a pure translation).

If $x = 0$, then $(0,F)$ is a Lorentz generator (symmorphic algebra) and $(x',0)$ generates a one-dimensional invariant subspace under $(0,F)$. Column 2 of Table IV.4 gives us the translational direction which could be chosen up to a conjugation under the stabilizer of $(0,F)$.

It will be useful to remark that, if x' is not light like, x and x' can be chosen to be orthogonal ; this is easily seen by choosing the two new generators, namely $(x+\alpha x',F)$ and $(x',0)$ with $\alpha = -\frac{x.x'}{x'^2}$.

If x' is light like, we have two cases to consider :

i) the couple (x,x') span a hyperbolic plane ; then by linear combination we can choose x along the second light like direction.

ii) The couple (x,x') span a tangent plane ; then x is space like and is defined up to a vector collinear to x'.

We give the corresponding results in the boxes "Abelian algebras" $\dim \mathcal{F}_A = 1$ of Table VIII.1 .

3. Case $F \neq 0$, $\lambda = 1$.

Following the relation (III.9) , F and F^* must constitute a basis of a two dimensional Lorentz algebra. Table IV.2 insures us that there exists two non conjugate such algebras, namely $SO(2) \times L(1,1)$ generated by J_z, K_z and $A(2)$ generated by A_x, A_y. (Let us recall: $A_x = -J_y + K_x$, $A_y = J_x + K_y$)

If F and F^* correspond to J_z and K_z respectively, then F and F^* are orthogonal tensors. The relation (VIII.3) is only possible if $F.x' = F^*.x = 0$. Using Table VII.1 and Theorem VII.2 , we can choose x and x' to be zero. So there is non nonsymmetric algebra of this type.

If F and F^* correspond to A_y and A_x respectively, the relation (II.3) is only possible if Fx' and F^*x are along the light like direction intersection of the F and F^* planes. So x and x' lie in the hyperplane generated by F and F^*. By an inner automorphism we can then choose (x,F) as one of the following elements of Table II.5 : A_y , $A_y + P_x$, $A_y - P_x$.

Let us note that after such a transformation A_x can always be kept as the $\mathcal{L}$- part of the second generator,

i) $(x,F) = A_y$

In this case, $Fx' = 0$ and x' lies in the F^* plane. From Table VII.1, a translation a which stabilizes A_y is such that $F^*.a$ is along the light like direction of the F^* plane or zero. So we can choose x' along the x-axis. Now a boost along the x-axis leads to the three non conjugate algebras:

$A(2)$ generated by A_x , A_y .

$A_+(2) = A_+(1) \times A(1)$ generated by $A_x + P_x$, A_y .

$A_-(2) = A_-(1) \times A(1)$ generated by $A_x - P_x$, A_y .

Note that $A_+(2)$ and $A_-(2)$ are conjugate to $A(1)$.

ii) $(x,F) = A_y + \varepsilon P_x$; $\varepsilon = \pm 1$

A similar discussion shows that x' is in the (x,y) plane. A boost along the z axis and the discrete transformation corresponding to a rotation of π around the y axis lead us to the following algebras:

$A^{\uparrow\downarrow}(2) = A^{\uparrow}(1) \times A^{\downarrow}(1)$ generated by $A_x + P_y$, $A_y + P_x$.

$A^{\uparrow\uparrow}_\alpha(2) = A^{\uparrow}_\alpha(1) \times A^{\uparrow}_\alpha(1)$ generated by $A_x + P_y$, $A_y + P_x + \alpha P_y$; $\alpha \neq 0$

Note that $A^{\uparrow}_\alpha(1)$ is conjugate to $A^{\uparrow}(1)$

B) - NON ABELIAN ALGEBRAS

Relations (VII.2) and (VIII.2) give

(VIII.5) $\quad F.x' - F'.x = x'$

(VIII.6) $\quad F \Delta F' = F'$

This last relation necessarily implies $F \neq 0$. Let us consider separately the two following cases.

1. Case $F' = 0$

With the aid of a conjugation, the generators (x,F) can be put as one of the elements of the first column of Table (II.5).

Moreover the relation (VIII.5) becomes :

(VIII.7) $\qquad F.x' = x'$.

So using §d of section VII , we only have to consider for (x,F) the three canonical elements J_ξ, K_z and $r K_z + P_y$, $r > 0$. Finally, we obtain the algebras

$T^0(1) \,\square\, F_\xi$ generated by $P_t - P_z$, J_ξ .

$T^0(1) \,\square\, L(1,1)$ generated by $P_t - P_z$, K_z .

$T^0(1) \,\square\, V_r$ generated by $P_t - P_z$, $r K_z + P_y$, $r > 0$.

2. Case $F' \neq 0$

Using once more Table IV.2 , we can choose as F and F' the generators K_z and A_x respectively. Relation (VIII.5) shows that x' belongs to the hyperplane generated by F and F' . Moreover, a suitable translational automorphism $(a,1)$ will transform x into $x_1 = x - F.a$ and x' into $x'_1 = x' - F'.a$, x'_1 corresponding to the light like direction belonging to the F plane and not in the F' plane. It follows that $Fx'_1 = -x'_1$.

From relation (VIII.5) we deduce that

$$x'_1 = 0$$

$$F'x_1 = 0 .$$

By a translation which stabilizes F' , we obtain the following types of algebras :

$S(1)$ generated by K_z, A_x .

$S_r(1) = V_r \,\square\, A(1)$ generated by $r K_z + P_y$, A_x .

dim	Symmorphic Algebras		Nonsymmorphic Algebras			
	ABELIAN ALGEBRAS (in ref [6] $(g_1)^2$; in ref [7] $2g_1$)					
2	$T^+(2)$	$\{P_t, P_z\}$				
	$T^0(2)$	$\{P_t - P_z, P_x\}$				
	$T^-(2)$	$\{P_x, P_y\}$				
1	* $T^+(1) \times SO(2)$	$\{P_t, J_z\}$	$T^+(1) \times H_\omega^+$	$\{P_z, \omega J_z + P_t\}$ $\omega > 0$	$T^-(1) \times A^-(1)$	$\{P_x, A_y\ P_t + P_z\}$
	$T^0(1) \times SO(2)$	$\{P_t - P_z, J_z\}$	$T^-(1) \times H_\omega^-$	$\{P_t, \omega J_z + P_z\}$ $\omega \neq 0$	$T^0(1) \times A^+(1)$	$\{P_t - P_z, A_y + P_x\}$
	* $T^-(1) \times SO(2)$	$\{P_z, J_z\}$	$T^0(1) \times H^+$	$\{P_t + P_z, J_z - (P_t - P_z)\}$	$T^0(1) \times A^+(1)$	$\{P_t - P_z, A_y - P_x\}$
	* $T^-(1) \times L(1,1)$	$\{P_x, K_z\}$	$T^0(1) \times H^+$	$\{P_t + P_z, J_z + (P_t - P_z)\}$		
	$T^0(1) \times A(1)$	$\{P_t - P_z, A_y\}$	$T^-(1) \times V_r$	$\{P_x, r K_z + P_y\}$ $r > 0$		
	* $T^-(1) \times A(1)$	$\{P_x, A_y\}$	* $T^0(1) \times A^-(1)$	$\{P_t - P_z, A_y + P_t + P_z\}$		
0	$SO(2) \times L(1,1)$	$\{J_z, K_z\}$	$A_+(2)$	$\{A_y, A_x + P_z\}$		
	$A(2)$	$\{A_x, A_y\}$	$A_-(2)$	$\{A_y, A_x - P_z\}$		
			$A^{tb}(2)$	$\{A_x + P_y, A_y + P_x\}$		
			$A_\alpha^{tb}(2)$	$\{A_x + P_z, A_y + P_x + \alpha P_y\}$ $\alpha \neq 0$		
	S(1) ALGEBRAS (in ref [6] & [7] : g_2)					
2						
1	$T^0(1) \,\square\, F_\xi$	$\{P_t - P_z, J_\xi\}$	$T^0(1) \,\square\, V_r$	$\{P_t - P_z, r K_z + P_y\}$ $r > 0$		
	$T^0(1) \,\square\, L(1,1)$	$\{P_t - P_z, K_z\}$				
0	$S(1)$	$\{K_z, A_x\}$	$S_r(1)$	$\{r K_z + P_y, A_x\}$ $r > 0$		

TABLE VIII.1 - TWO DIMENSIONAL SUBALGEBRAS IN

In this table the subgroups which are preceded by * are stabilizers of coadjoint orbits of $\mathcal{P}$ (ref. [8]

IX - THREE DIMENSIONAL LIE SUBALGEBRAS

Classification of the three-dimensional subalgebras, up to a conjugation can be obtained in the same way as the classification of two-dimensional Lie subalgebras. First, let us recall the eleven three-dimensional real Lie algebras, up to an isomorphism [2][6][7]. In Table IX.1, we note $\{e,f,g\}$ a basis of a three-dimensional Lie algebra and by $[\,,\,]$ the Lie bracket.

	$[e,f]=$	$[e,g]=$	$[f,g]=$	Notations of Ref [6]	Notations of Ref [7]
Abelian algebra T(3)	0	0	0	$(g_1)^3$	$3g_1$
Weyl algebra W(1)	g	0	0	$g_{3,1}$	$g_{3,1}$
T(1) x S(1)	f	0	0	$g_1 \times g_2$	g_2+g_1
$Th_\lambda(2)$ $\lvert\lambda\rvert>1$	λf	g	0	$g_{3,5}(\lambda)\ ;\lvert\lambda\rvert>1$	$g_{3,5}(\frac{1}{\lambda})\ ;\lvert\lambda\rvert>1$
E(1,1)	$-f$	g	0	$g_{3,6}(-1)$	$g_{3,4}(-1)$
Th(2)	f	g	0	$g_{3,6}(1)$	$g_{3,3}$
Jor(2)	f	$g+f$	0	$g_{3,7}$	$g_{3,2}$
$S_\alpha(2)$ $(\alpha>0)$	$\alpha_1 f+\alpha_2 g$	$-\alpha_2 f+\alpha_1 g$	0	$g_{3,8}(\alpha)\ ;\alpha>0$	$g_{3,5}(\alpha)\ ;\alpha>0$
E(2)	g	$-f$	0	$g_{3,9}(0)$	$g_{3,5}(0)$
SO(3)	g	$-f$	e		$g_{3,7}$
SO(2,1)	g	$-f$	$-e$		$g_{3,6}$

TABLE IX.1 - THREE DIMENSIONAL REAL ALGEBRAS

A) - ABELIAN ALGEBRAS

Let $\mathcal{A} = \mathcal{T}_A + \mathcal{L}_A + \mathcal{M}_A$ be such an algebra, we can study these algebras following the dimension of $\mathcal{T}_A$. Let us note that $\dim(\mathcal{L}_A + \mathcal{L}'_A) = \dim \mathcal{A} - \dim \mathcal{T}_A = 3 - \dim \mathcal{T}_A$.

By a suitable L - conjugation we can put the algebra $\mathcal{L}_A + \mathcal{L}'_A$ in the "canonical" form, that is to say corresponding to a basis given in the first column of Table IV.4.

Since there exists no Abelian Lorentz three-dimensional subalgebras we only have to consider the cases where $1 \leq \dim \mathcal{T}_A \leq 3$.

i) $\dim \mathcal{T}_A = 1$

The allowed cases for $\mathcal{L}_A + \mathcal{L}'_A$ are SO(2) x L(1,1) and A(2) generated by J_z, K_z and A_x, A_z respectively. The form of $\mathcal{T}_A$ is then given in the second column of the same table. Because of the Abelianity, we immediately note that the case corresponding to $\mathcal{L}_A + \mathcal{L}'_A = A(2)$ and $\mathcal{T}_A$ generated by $P_y - P_z$ is the only one to be considered. It is then interesting to remark that in this last case $\mathcal{A}$ can be written as the direct product $\mathcal{T}_A \times (\mathcal{L}_A + \mathcal{M}_A)$. Moreover, Table IV.3 gives us the stabilizers of the algebra A(2), and we can see that this stabilizer S(2) stabilizes also $P_y - P_z$. Consequently $\mathcal{L}_A + \mathcal{M}_A$ can be chosen among the two-dimensional subalgebras given in Table VIII.1. The obtained algebras are given in Table IX.2 (Abelian algebras, $\dim \mathcal{T}_A = 1$).

ii) $\dim \mathcal{T}_A = 2$

$\mathcal{L}_A + \mathcal{L}'_A$ can be chosen as a one-dimensional Lie algebra given in Table IV.4. The same table gives us the two-dimensional translation algebras invariant under one $\mathcal{L}$-generatorβ. Because of Abelianity and using conjugation we find the algebras given in Table (IX.2) (Abelian algebra, $\dim \mathcal{T}_A = 2$).

iii) $\dim \mathcal{T}_A = 3$

Finally we note that the case $\dim \mathcal{T}_A = 3$ has been given in Table IV.1 . We recall those results in Table IX.2 .

B) - W(1) ALGEBRAS

Classification of Lorentz algebras shows that there exist no three-dimensional subalgebras isomorphic to W(1). So we are left with the cases where $1 \leq \dim \mathcal{T}_A \leq 2$, a pure translational algebra being necessarily Abelian.

i) $\dim \mathcal{T}_A = 1$

From commutation relations given in Table IX.1 , we can easily deduce that Q_γ generates $\mathcal{T}_A$, and that $\mathcal{L}_A + \mathcal{L}'_A$ is Abelian. Moreover by using Table IV.4 , that $\mathcal{L}_A + \mathcal{L}'_A$ is the A(2) algebra and $\mathcal{T}_A$ generated by $P_\varepsilon - P_z$.

Translation and conjugation of the stabilizer of A(2) allows us to find the non conjugate algebras given in Table IX.2 (W(1) - algebras, $\dim \mathcal{T}_A = 1$).

ii) $\dim \mathcal{T}_A = 2$

Let e be the non translational generator and write

$$e = (x, F) \quad ; \quad f = (y, 0) \quad ; \quad g = (z, 0) \ .$$

Commutation relations give

$$\text{(IX.1)} \qquad \begin{cases} F.y = z \\ F.z = 0 \end{cases}$$

So F is a plane tensor, and x belongs to the intersection of the F plane and F^* plane. It follows, from Table VII.1 , that F corresponds to an element conjugate to A_y . Then g is proportional to $P_t - P_z$. Because the stabilizer of A_y also stabilizes $P_t - P_z$, e can be chosen in Table II.5 . We finally obtain the algebras given in Table IX.2 (W(1) algebras, $\dim \mathcal{T}_A = 2$).

C) - T(1) x S(1) ALGEBRAS

First let us remark, as in the previous case, that $1 \leq \dim \mathcal{T}_A \leq 2$. Moreover, it is always possible, by an inner automorphism, to transform the S(1) subalgebra into one of the subalgebras written in the second part of Table VIII.1 . Hence it will be easy to choose the generator of T(1). which must commute with S(1) . More precisely,

i) $\dim \mathcal{T}_A = 1$

Two cases are possible. By the same way we denote $\mathcal{L}_A$, $\mathcal{L}'_A$, $\mathcal{T}_A$, $\mathcal{T}'_A$, $\mathcal{M}_A$ the different parts of the subalgebra $\mathcal{A}$, let us denote $\mathcal{L}_{S(1)}$, $\mathcal{L}'_{S(1)}$, $\mathcal{T}_{S(1)}$, $\mathcal{T}'_{S(1)}$, $\mathcal{M}_{S(1)}$, the different parts of the subalgebra isomorphic to S(1) which is included in $\mathcal{A}$. Two cases are possible:

a) $\dim \mathcal{T}_{S(1)} = 0$. Then $\mathcal{L}_{S(1)} + \mathcal{L}'_{S(1)} = \mathcal{L}_A + \mathcal{L}'_A$ and can be generated by K_z, A_x . It follows that T(1) must be generated by P_y and we obtain only a type of algebra $T^-(1) \times S(1)$.

b) $\dim \mathcal{T}_{S(1)} = 1$. Then $\mathcal{L}_{S(1)} + \mathcal{L}'_{S(1)}$ is generated by K_z or J_z .

In this case T(1) cannot be a translational algebra and we obtain the algebra $T^*(1) \oplus [SO(2) \times L(1,1)]$ generated by $P_t - P_z, J_z, K_z$.

ii) $\dim \mathcal{T}_A = 2$.

The only generator of $\mathcal{L}_A + \mathcal{L}'_A$ is K_z , and we find the algebras given in Table IX.2 (T(1) x S(1) algebras, $\dim \mathcal{T}_A = 2$).

D) - $Th_\lambda(2)$, $|\lambda| > 1$ ALGEBRAS

A simple discussion can show us that there exists no subalgebra of this type in the Poincaré algebra. As there exists no such algebra in $\mathcal{T}(\mathcal{L})$ we only have to consider $1 \leq \dim \mathcal{T}_A \leq 2$.

Let us remark that such an algebra contains two algebras $S(1)_1$ and $S(1)_2$ isomorphic to S(1) and generated by e,f and e,g respectively.

i) $\dim \mathcal{T}_A = 1$

We could choose $\mathcal{L}_{S(1)_1} + \mathcal{L}'_{S(1)_1}$ generated by K_z and A_x. But there exists no translational element g satisfying commutation relations.

iii) $\dim \mathcal{T}_A = 2$

We could choose $\mathcal{L}_A + \mathcal{L}'_A$ generated by K_z. But there do no exist translational elements f and g satisfying the commutation relations.

E) - E(1,1) ALGEBRAS

Similar considerations as those of case D allow us to restrict ourselves to $\dim \mathcal{T}_A = 2$, and $\mathcal{L}_A + \mathcal{L}'_A$ generated by K_z or J_z. These algebras are given in Table IX.2 .

F) - Th(2) ALGEBRAS

There exists, in $\Pi(\mathcal{L})$, an algebra isomorphic to Th(2) ; so we have to consider $0 \leq \dim \mathcal{T}_A \leq 2$.

i) $\dim \mathcal{T}_A = 0$

By a L-conjugation we can choose $\mathcal{L}_A + \mathcal{L}'_A$ generated by K_z, A_x, A_y. Let us note $f = (x,F)$, $g = (y,F^*)$ and $e = (z,F')$ with F and F' corresponding to A_x and K_z respectively. By action of a T-conjugation and by solving the system

(IX.2)
$$\begin{cases} F.y - F^*.x = 0 \\ F'.x - F.z = x \\ F'.y - F^*.z = y \end{cases}$$

we obtain one algebra, namely Th(2) generated by K_z, A_x, A_y.

ii) $\dim \mathcal{T}_A = 1$

Such an algebra contains two subalgebras $S(1)_1$ and $S(1)_2$ isomorphic to S(1) generated by e,f and e,g. We could take $S(1)_1$ generated by K_z, A_x or by $r K_z + P_z, A_x$. By action of a T conjugation and by solving the equations :

(IX.3)
$$\begin{cases} F.y = 0 \\ F'.x - F.z = x \\ F'.y = y \end{cases}$$

We obtain the two types of algebras given in Table IX.2 (Th(2) algebras, $\dim \mathcal{T}_A = 1$).

iii) $\dim \mathcal{T}_A = 2$

The equations

(IX.4)
$$\begin{cases} F'.x = x \\ F'.y = y \end{cases}$$

show that there exists no subalgebra of this type.

G) - Jor(2) ALGEBRAS

We remark once more that $1 \leq \dim \mathcal{T}_A \leq 2$ and moreover that e cannot be a pure translational generator and that f must be a $\mathcal{T}$-element (if not, g would be a $\mathcal{T}$-element and the second commutation relation could not hold). Since e and f generate an algebra isomorphic to S(1), we deduce after conjugation, that $\mathcal{L}_{S(1)} + \mathcal{L}'_{S(1)}$ is generated by K_z or J_z and f is proportional to $P_t - P_z$. This last remark prohibides g to be a $\hat{\mathcal{T}}$-element. We finally obtain the two types given in Table IX.2 .

H) - Jor(2) (sin $\alpha \neq 0$) ALGEBRAS

Let us write $e = (x_1,F_1)$, $f = (x_2,F_2)$ and $g = (x_3,F_3)$.

(IX.5)

It is clear from commutation relations that $F_1 \neq 0$ and that $\dim \mathcal{T}_A \neq 1$.

Moreover solving the equations

$$(IX.5) \qquad \begin{cases} F_1 . x_2 = \sin\frac{\alpha}{\ell}\, x_2 + \cos\frac{\alpha}{\ell}\, x_3 \\ F_1 . x_3 = -\cos\frac{\alpha}{\ell}\, x_2 + \sin\frac{\alpha}{\ell}\, x_3 \end{cases}$$

we deduce that x_1 and x_2 must be zero.

Finally we have to look at the case $\dim \mathcal{T}_A = 0$. In this case we can choose $\mathcal{L}_A + \mathcal{L}'_A$ generated by J_ℓ, A_x, A_y.

By a translation we can put $x_3 = 0$ and equations (IX.5) insure us that we have only the symmorphic algebra $S_\alpha(2)$ generated by J_α, A_x, A_y.

I) - E(2) ALGEBRAS

Commutation relations show us that $\dim \mathcal{T}_A$ can be 0 or 2.

i) $\dim \mathcal{T}_A = 0$

We can choose $\mathcal{L}_A + \mathcal{L}'_A$ generated by J_z, A_x, A_y.

Writing $f = (x,F)$, $g = (y,F^*)$ and $e = (z,F')$ with F and F' corresponding to A_x and J_z respectively, we have to solve the equations

$$(IX.6) \qquad \begin{cases} F.y - F^*.x = 0 \\ F'.x - F.z = y \\ F'.y - F^*.z = -x \end{cases}$$

Using translational conjugations, we obtain up to a boost automorphism three types of algebras given in Table IX.1.

ii) $\dim \mathcal{T}_A = 2$

We can always choose $\mathcal{L}_A + \mathcal{L}'_A$ generated by J_z or J_ℓ. By a conjugation $e = (z,F')$ can be taken in Table II.5. Now using the two-dimensional invariant subspaces given in Table IV.4, we obtain the algebras given in Table IX.2 (E(2) algebras, $\dim \mathcal{T}_A = 2$).

J) - SO(3) ALGEBRAS

We first note that a three-dimensional subalgebra $\mathcal{A}$ of an algebra $\mathcal{C} \oplus \mathcal{T}$, with $\mathcal{T}$ Abelian, such that it is equal to its derived algebra $\mathcal{A}' = \mathcal{A}$ cannot contain $\mathcal{T}$-elements. It is clear that in our case, $\mathcal{L}_A + \mathcal{L}'_A$ can be chosen as generated by J_x, J_y, J_z. It follows that $\mathcal{A}$ is a subalgebra of $A_2^+(3)$ generated by $\vec{J}, \vec{P}, P_0$.

Noting that $A_2^+(3)$ can be written as $E(3) \times R$, we deduce that the algebra $\mathcal{A}$ under consideration is a subalgebra of $E(3)$. This shows us that we have only one algebra, up to a conjugation, isomorphic to SO(3), namely the one generated by J_x, J_y, J_z.

K) - SO(2,1) ALGEBRAS

By the same arguments where $A_2^+(3)$ is replaced by $A_2^-(2,1)$ and E(3) by E(2,1), we obtain the algebra ~~E(2,1)~~ SO(2,1) generated by J_y, K_x, K_z.

All these results are gathered in Table IX.2. This Table is divided into subtables, each of them corresponding to an isomorphism class. Inside each subtable, subalgebras are classified in symmorphic and non-symmorphic ones and according to the dimension of $\mathcal{T}_A$.

dim ℰ	Isomorphic Algebras		Non-isomorphic Algebras		

[illegible] ALGEBRAS.

[illegible]	[illegible]	[illegible]			
	[illegible]	[illegible]			
	[illegible]	[illegible]			
2	[illegible]	[illegible]			
	[illegible]	[illegible]			
	[illegible]	[illegible]	[illegible] $\times A(1)$	$\{P_t - P_z, [illegible], A_x + P_t + P_z\}$	
1	[illegible]	[illegible]	[illegible] A_z [illegible]	$\{P_t - P_z, [illegible], A_y\}$	$\ne \pm 1$
			[illegible] $\times A^{2b}(2)$	$\{P_t - P_z, A_x + P_y, A_y + P_x\}$	
			[illegible] $\times A_\alpha^{2b}(2)$	$\{P_t - P_z, A_x + P_y, A_y + P_z + \alpha P_y\}$	$\alpha \ne 0$

W(2) [illegible]GEBRAS

2	[illegible]	[illegible]	[illegible]	$\{P_t - P_z, [illegible], A_y + P_x\}$	
	[illegible]	[illegible]	[illegible]	$\{P_t - P_z, [illegible], A_y - P_x\}$	
			[illegible]	$\{P_t - P_z, [illegible], A_y + P_t + P_z\}$	
			[illegible] $\circ A^{\alpha}(1)$	$\{P_t - P_z, P_y + \alpha P_x, A_y + P_x\}$	
			[illegible] $\circ A^{\alpha}(1)$	$\{P_t - P_z, P_y + \alpha P_x, A_y - P_x\}$	[illegible]
			[illegible]	$\{P_t - P_z, P_y - \alpha P_x, A_y + P_t + P_z\}$	
1			[illegible] $\times A^b(1) \circ A(1)$	$\{P_t - [illegible], A_x, A_y - P_x\}$	
			[illegible] $\times A^b(1)] \circ A(1)$	$\{P_t - P_z, A_x, A_y + P_x\}$	
			[illegible] $\times A_\alpha^b(1)] \circ A(1)$	$\{P_t - P_z, [illegible], A_y - P_x + \alpha P_y\}$	$\alpha \ne 0$
			[illegible] $\times A_\alpha^b(1)] \circ A(1)$	$\{P_t - P_z, A_x, A_y + P_x + \alpha P_y\}$	$\alpha \ne 0$
			[illegible] $\times A^b(1)] \circ A^{\alpha}(1)$	$\{P_t - P_z, A_x + P_y, A_y + \alpha P_x\}$	$\alpha \ne 0, \pm 1$
			[illegible] $\times A^b(1)] \circ A^{1}(1)$	$\{P_t - P_z, A_x - P_y, A_y - P_x\}$	
			$[T^0(1) \times A^b(1)] \circ A^{\alpha}(1)$	$\{P_t - P_z, A_x - P_y, A_y + \alpha P_x\}$	$\alpha \ne 0, \pm 1$
			[illegible] $\times A^b(1)] \circ A^{1}(1)$	$\{P_t - P_z, A_x - P_y, A_y + P_x\}$	
			$[T^0(1) \times A^b(1)] \circ A^{(\alpha,\beta)}(1)$	$\{P_t - P_z, A_x + P_y, A_y + \alpha P_x + \beta P_y\}$	$\alpha.\beta \ne 0, \alpha \ne +1$
			$[T^0(1) \times A^b(1)] \circ A^{(\alpha,\beta)}(1)$	$\{P_t - P_z, A_x - P_y, A_y + \alpha P_x + \beta P_y\}$	$\alpha.\beta \ne 0, \alpha \ne -1$

TABLE IX.2 - THREE-DIMENSIONAL SUBALGEBRAS IN $\mathcal{P}$ (Beginning)

$T(1) \times S(1)$ ALGEBRAS

2	$T^0(2) \circ L(1,1)$	$\{P_t - P_z, P_x, K_z\}$	$T^0(2) \circ V_\sigma$	$\{P_t - P_z, P_x, \sigma K_z + P_y\}$	$\sigma > 0$
[illegible]	$T^-(1) \times S(1)$	$\{P_y, K_z, A_x\}$			
	$T^-(1) \circ [So(2) \times L(1,1)]$	$\{P_t - P_z, J_z, K_z\}$			

$E(1,1)$ ALGEBRAS

2	$T^-(2) \circ F_\rho$	$\{P_t, P_z, J_z\}$	$T^+(2) \circ V_\sigma$	$\{P_y, P_z, \sigma K_z + P_x\}$	$\sigma > 0$
	$T^+(2) \circ L(1,1)$	$\{P_t, P_z, K_z\}$			

$Jor(2)$ ALGEBRAS

1			$Jor_\sigma(2)$	$\{P_t - P_z, A_x, \sigma K_z + P_y\}$	$\sigma > 0$
			$Jor_{\sigma,\alpha}(2)$	$\{P_t - P_z, A_x, \sigma K_z + P_x + \alpha P_y\}$	$\sigma > 0$, $\alpha \ne 0$

$Th(2)$ ALGEBRAS

1	$T^0(1) \circ S(1)$	$\{P_t - P_z, P_y, A_x\}$	$T^0(1) \circ S_\sigma(1)$	$\{P_t - P_z, \sigma K_z + P_y, A_x\}$	$\sigma > 0$
0	$Th(2)$	$\{K_z, A_x, A_y\}$			

$S_t(2)$ ALGEBRAS

0	$S_t(2)$	$\{K_z, A_x, A_y\}$			

$E(2)$ ALGEBRAS

2	$T^-(2) \circ So(2)$	$\{P_x, P_y, J_z\}$	$T^-(2) \circ H_\omega^+$	$\{P_x, P_y, \omega J_z + P_t\}$	$\omega > 0$
	$T^-(2) \circ F_\rho$	$\{P_x, P_y, J_z\}$	$T^-(2) \circ H_\omega^-$	$\{P_x, P_y, \omega J_z + P_t\}$	$\omega \ne 0$
			$T^-(2) \circ H^\uparrow$	$\{P_x, P_y, J_z - (P_t - P_z)\}$	
			$T^-(2) \circ H^\downarrow$	$\{P_x, P_y, J_z + (P_t - P_z)\}$	
0	$E(2)$	$\{J_z, A_x, A_y\}$	$A(2) \circ H^\uparrow$	$\{A_x, A_y, J_z - (P_t - P_z)\}$	
			$A(2) \circ H^\downarrow$	$\{A_x, A_y, J_z + (P_t - P_z)\}$	

$So(3)$ ALGEBRAS

0	$So(3)$	$\{J_x, J_y, J_z\}$			

$So(2,1)$ ALGEBRAS

0	$L(2,1)$	$\{J_z, K_x, K_y\}$			

TABLE IX.2 - THREE-DIMENSIONAL SUBALGEBRAS IN $\mathcal{P}$ (end)

K - FOUR, FIVE AND SIX DIMENSIONAL LIE SUBALGEBRAS OF

The search for the subalgebras of dimension 4 , 5 and 6 will be conducted in a different way. The reason is the great number of isomorphism classes of 4-dimension [6] [7], and a fortiori of 5 and 6-dimension algebras. Let us add that the complete classification of 5 and 6 dimensional algebras is not known to us, but the relatively small number of Poincaré subalgebras of these dimensions do not require the knowledge of this classification. In the method used in the following, the subalgebras will be first classified up to a conjugation, before to be gathered in isomorphism classes.

The last problem is the classification of these algebras up to an isomorphism. For four dimensional algebras we can compare the non conjugate subalgebras with the classification given in Table X.1 .

The final results concerning the four-dimensional algebras are given in Table X.2

For five and six-dimensional algebras, we will use a simple property on isomorphic algebras. Let us consider, for an algebra $\mathcal{A}$, the derived series

$$\mathcal{A} = D^0\mathcal{A} \supseteq D^1\mathcal{A} \supseteq \cdots D^k\mathcal{A} \supseteq \cdots$$

with

$$D^k\mathcal{A} = \left[D^{k-1}\mathcal{A}, D^{k-1}\mathcal{A}\right]$$

and the lower central series

$$\mathcal{A} = C^0\mathcal{A} \supseteq C^1\mathcal{A} \supseteq \cdots C^k\mathcal{A} \supseteq \cdots$$

with

$$C^k\mathcal{A} = \left[C^{k-1}\mathcal{A}, \mathcal{A}\right] .$$

It is easy to remark that if $\mathcal{A}$ and $\mathcal{B}$ are two isomorphic algebras, then $D^k\mathcal{A}$ and $D^k\mathcal{B}$. are isomorphic, so are isomorphic $C^k\mathcal{A}$ and $C^k\mathcal{B}$.

In consequence, we will be sure that two algebras $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic if there exists $k \in \mathbb{N}$ such that $\dim D^k\mathcal{A} \neq \dim D^k\mathcal{B}$. or $\dim C^k\mathcal{A} \neq \dim C^k\mathcal{B}$. A more precise study will have to be done in the contrary case ; we just give the following example.

Let us look the two algebras $\mathcal{A}$ generated by K_2, A_x, A_y $P_y, P_t - P_z$ and $\mathcal{B}$ generated by $cK_2 + P_y$, $A_x, A_y, P_y, P_t - P_z$. For each k , $D^k\mathcal{A}$ and $D^k\mathcal{B}$ are same dimensional, so are $C^k\mathcal{A}$ an , and $C^k\mathcal{B}$. However in $\mathcal{A}$, $\mathrm{ad}K_2$ induces in $D^1\mathcal{A}$ the identity an and there exists no element of $\mathcal{B}$ which induces in $D^1\mathcal{B}$ the identity. So these two algebras are not isomorphic.

We can sum up the classification up to a conjugation in remarking that $\mathcal{A} = \mathcal{C}_A + \mathcal{L}_A + \mathcal{M}_A$ being a subalgebra, the dimensions of $\mathcal{C}_A$ and $\mathcal{L}_A + \mathcal{L}'_A$ do not change by action of an inner automorphism. Moreover, the algebra $\mathcal{L}_A + \mathcal{L}'_A$ can be put into a "canonical form", that is to say, we can choose $\mathcal{L}_A + \mathcal{L}'_A$ as one of the algebras given in the first column of Table IV.4 . The other columns of this same Table furnish the possible $\mathcal{C}_A$ for $\mathcal{A}$. So we have for each dimension of $\mathcal{A}$ to research successively subalgebras $\mathcal{A}$ such that $\dim \mathcal{C}_A = 0, 1, \ldots, 4$, the corresponding dimension of $\mathcal{L}_A + \mathcal{L}'_A$ being $\dim \mathcal{A} - \dim \mathcal{C}_A$.

In the following, we do not give the tedious calculations which consist particularly to use Table IV.4 and we do not explicit the inner automorphisms which insure us that the algebras given in the following Tables are non conjugate

We give the results for five and six-dimensional subalgebras in Table X.3 and X.4 , respectively.

	$[e_1,\cdot]$	$[e_2,\cdot]$	$[e_3,\cdot]$	$[e_4,\cdot]$	$[e_1,e_2]$	$[e_3,e_4]$
* Abelian Algebra $T(4)$	0	0	0	0	0	0
* $T(1) \times W(1)$	g	0	0	0	0	0
* $T(2) \times S(1)$	f	0	0	0	0	0
* $g_{(4,1)}$	0	g	h	h	0	0
* $S(2)$	0	g	h	$-h$	g	0
* $g_{4,3}$	g	h	0	0	0	0
$g_{4,4}$	g	0	h	0	0	0
$S(1) \times S(1)$	f	0	0	0	0	h
$T(1) \times Th_\lambda(2)$; $\lvert\lambda\rvert>1$	λf	g	0	0	0	0
* $T(1) \times E(1,1) = A_2(1,1)$	$-f$	g	0	0	0	0
* $T(1) \times Th(2)$	f	g	0	0	0	0
* $T(1) \times J_{0,1}(2)$	f	$g+f$	0	0	0	0
$T(1) \times S_\alpha(2)$ $(\alpha\neq 0)$	$\alpha f + \omega_2 g$	$-\omega_2 f + \alpha g$	0	0	0	0
* $A_2(2) = T(2) \times E(2)$	g	$-f$	0	0	0	0
* $T(1) \times SO(3)$	g	$-f$	0	e	0	0
* $T(1) \times SO(2,1)$	g	$-f$	0	$-e$	0	0
$Th_{\alpha,\beta}(3)$ $-1<\alpha\leq\beta<0$. $-1\leq\alpha<0$; $0<\beta\leq 1$. $0\leq\alpha\leq\beta\leq 1$; $\alpha\beta\neq 1$.	f	αg	βh	0	0	0
* $Th(3)$	f	g	h	0	0	0
* $g_{4,5}(\alpha)$ $\alpha\neq 0$	αf	$g+h$	h	0	0	0
$J_{0,1}(3)$	$f+g$	$g+h$	h	0	0	0
* $g_{4,6}(\alpha,\beta)$ $\alpha>0$	αf	$\beta g - h$	$g+\beta h$	0	0	0
$g_{4,7}(\alpha)$ $0<\alpha\leq 2$, $\alpha\neq 1$	$(\alpha-1) f$	g	αh	h	0	0
$Osc(0,1)$	$-f$	g	0	h	0	0
$g_{4,10}$	$f+g$	g	$2h$	h	0	0
$g_{4,11}(\alpha)$; $\alpha>0$	$\alpha f - g$	$f+\alpha g$	$2\alpha h$	h	0	0
* $Osc(1,0)$	$-g$	f	0	h	0	0

TABLE X.1 - FOUR-DIMENSIONAL NON ISOMORPHIC REAL LIE ALGEBRAS

(Beginning)

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TABLE X.1 - FOUR-DIMENSIONAL NON ISOMORPHIC REAL LIE ALGEBRAS

(End)

"Let us note that the symbols $g_{4,3}$, $g_{4,4}$, $g_{4,6}(\alpha)$, $g_{4,8}(\alpha,\beta)$, $g_{4,9}(\alpha)$ $g_{4,10}$ and $g_{4,11}(\alpha)$ in the previous table have been taken in ref. (6) We have noted in the same table by * the types of algebras appearing in $\bar{P}$,

TABLE X.2 - FOUR-DIMENSIONAL ALGEBRAS IN $\mathcal{P}$

$\dim \mathfrak{G}_A$	Symmorphic Algebras	Non Symmorphic Algebras
	ABELIAN ALGEBRAS	
4	$T(4)$: $\{P_t, P_x, P_y, P_z\}$	
	$T(1) \times W(1)$ ALGEBRAS	
3	$T^0(3) \oplus A(1)$ $\{P_x, P_y, P_t - P_z, A_3\}$	$T^0(3) \oplus A^-(1)$ $\{P_x, P_y, P_t - P_z, A_3 + P_t + P_z\}$
2	$T^0(2) \oplus A(2)$ $\{P_3, P_t - P_z, A_x, A_y\}$	$[T^0(2) \oplus A^0(1)] \oplus A(1)$ $\{P_3, P_t - P_z, A_3 + P_x, A_x\}$ $[T^0(2) \oplus A^+(1)] \oplus A(1)$ $\{P_3, P_t - P_z, A_3 - P_z, A_x\}$
	$T(2) \times S(1)$ ALGEBRAS	
3	$T^0(2) \oplus L(1)$ $\{P_x, P_3, P_t - P_z, K_z\}$	

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$A_2(4,1) = T(1) \times E(4,1)$ ALGEBRAS

3	$A_2^-(4,1)$	$\{P_t, P_x, P_z, K_z\}$	$A_{2,\sigma}^-(4,1)$	$\{P_t, P_x, P_z, \sigma K_z + P_y\}$ $\sigma > 0$
2	$E(3,1) \times SO(2)$	$\{P_t, P_z, K_z, J_z\}$		

$T(1) \times Inv_\sigma(2)$ ALGEBRAS

2			$T^-(1) \times Inv_\sigma(2)$	$\{P_y, P_t - P_z, A_x, \sigma K_z + P_x\}$ $\sigma > 0$

$T(1) \times T_2^0(2)$ ALGEBRAS

2	$[T^+(1) \oplus S(1)] \times T^-(1)$	$\{P_y, P_t - P_z, A_x, K_z\}$	

$A_2(2) = T(1) \times E(2)$ ALGEBRAS

3	$T^0(3) \oplus SO(2)$	$\{K_z, P_y, P_t - P_z, J_z\}$	$T^0(3) \oplus H^\uparrow$	$\{P_x, P_y, P_t + P_z, J_z - (P_t - P_z)\}$
	$T^+(3) \oplus SG(2)$	$\{P_t, P_x, P_y, J_z\}$	$T^0(3) \oplus H^\downarrow$	$\{P_x, P_y, P_t + P_z, J_z + (P_t - P_z)\}$
	$T^-(3) \oplus SO(2)$	$\{P_x, P_y, P_z, J_z\}$	$T^+(3) \oplus H_\omega^+$	$\{P_t, P_x, P_y, \omega J_z + P_z\}$ $\omega \neq 0$
			$T^-(3) \oplus H_\omega^-$	$\{P_x, P_y, P_z, \omega J_z + P_t\}$ $\omega > 0$
2	$T^-(2) \oplus [SO(2) \times L(1,1)]$	$\{P_x, P_y, J_z, K_z\}$		
1	$T^0(1) \times E(2)$	$\{P_t - P_z, A_x, A_y, J_z\}$		

$T(1) \times SO(3)$ ALGEBRAS

1	$T^+(1) \times SO(3)$	$\{P_t, J_x, J_y, J_z\}$	

$T(1) \times SO(2,1)$ ALGEBRAS

1	$T^-(1) \times SO(2,1)$	$\{P_y, J_y, K_x, K_z\}$	

$\mathcal{G}_{4,1}$ ALGEBRAS

2	$T^0(2) \oplus S(1)$	$\{P_x, P_t - P_z, A_x, K_z\}$	$T^0(2) \oplus S_\sigma(1)$	$\{P_x, P_t - P_z, A_x, \sigma K_z + P_y\}$ $\sigma > 0$
	$T_\alpha^0(2) \oplus S(1)$	$\{P_x + \alpha P_y, P_t - P_z, A_x, K_z\}$ $\alpha \neq 0$	$T_\alpha^0(2) \oplus S_\sigma(1)$	$\{P_x + \alpha P_y, P_t - P_z, A_x, \sigma K_z + P_y\}$ $\sigma > 0$ $\alpha \neq 0$

TABLE X.2 - FOUR-DIMENSIONAL ALGEBRAS IN $\mathcal{G}$

(Continued)

$S(2)$ ALGEBRAS

0	$S(2)$	$\{J_z, K_x, A_x, A_y\}$	

$\mathcal{G}_{4,3}$ ALGEBRAS

3	$T^+(2) \oplus A(1)$	$\{P_t, P_y, P_z, A_y\}$	$T^+(2) \oplus A^\uparrow(1)$	$\{P_t, P_y, P_z, A_y + P_x\}$
			$T^+(2) \oplus A^\downarrow(1)$	$\{P_t, P_y, P_z, A_y - P_x\}$
2			$[T^0(2) \times A^-(1)] \oplus A(1)$	$\{P_x, P_t - P_z, A_x, A_y - P_t + P_z\}$
			$[T^0(2) \times A^-(1)] \oplus A^\gamma(1)$	$\{P_x, P_t - P_z, A_x + \alpha P_y, A_y + P_t + P_z\}$ $\alpha \neq 0$

$T_2^0(2)$ ALGEBRAS

1	$T_2^0(2)$	$\{P_t - P_z, K_z, A_x, A_y\}$	

$\mathcal{G}_{4,5}(\gamma)$ ALGEBRAS

1		$[T^0(1) \times A(2)] \oplus V_\sigma$	$\{P_t - P_z, A_x, A_y, \sigma K_z + P_x\}$ $\sigma > 0$

$\mathcal{G}_{4,5}(1,0)$ ALGEBRAS

3	$T^0(3) \oplus F_\beta$	$\{P_x, P_y, P_t - P_z, J_\beta\}$	

$\mathcal{G}_{4,7}(\alpha,\alpha)$ ALGEBRAS

1	$[T^0(1) \times A(2)] \oplus F_\beta$	$\{P_t - P_z, A_x, A_y, J_z\}$	

$Goe(1,0)$ ALGEBRAS

1		$Goe(1,0)^\uparrow$	$\{P_t - P_z, A_x - P_y, A_y + P_x, J_z\}$
		$Goe(1,0)^\downarrow$	$\{P_t - P_z, A_x + P_y, A_y - P_x, J_z\}$

TABLE X.2 - FOUR-DIMENSIONAL ALGEBRAS IN $\mathcal{G}$

(End)

In this table the subgroups which are preceded by $\star$ are stabilizers of coadjoint orbits of $\mathcal{G}$ (ref [8]).

dim	Isomorphic Algebras	Non-isomorphic Algebras
	$W(2)$	$[e,f] = [g,h] = k$
3	$W(2)$ $\{P_x, P_y, P_t - P_z, A_x, A_y\}$	
	$T(2) \times F(1,1)$	$[e,f] = -f$; $[e,g] = g$
3	$T(2) \times L(1,1)$ $\{P_t, P_x, P_y, P_z, K_z\}$	
	$T(2) \times E(2)$	$[e,f] = g$; $[e,g] = -f$
4	$T(4) \circ S(2)$ $\{P_t, P_x, P_y, P_z, J_z\}$	
	$T(1) \times g_{4,1}$	$[e,g] = g$; $[e,h] = h$, $[f,g] = k$
3	$T^{s}(1) \circ S(1)$ $\{P_x, P_y, P_t - P_z, A_x, K_z\}$	
	$T(1) \times g_{4,3}$	$[e,f] = g$, $[e,g] = h$
4	$T(4) \circ A(1)$ $\{P_x, P_y, P_z, P_t, A_z\}$	
	$g_{5,1}$	$[e,f] = g$; $[e,g] = k$; $[f,h] = k$
3		$\{P_x, P_y, P_t - P_z, A_x + (P_t + P_z), A_y\}$
	$E(2) \times S(1)$	$[e,f] = g$; $[e,g] = -f$, $[h,k] = k$
	$T \cdot [SO(2) \times L(1,1)]$ $\{P_x, P_y, P_t - P_z, J_z, K_z\}$	
		$[e,g] = h$; $[e,h] = -g$; $[f,g] = g$; $[f,h] = h$; $[f,k] = k$
1	$T^{s}(1) \circ S(2)$ $\{P_t - P_z, A_x, A_y, J_z, K_z\}$	
		$[e,f] = f$; $[e,g] = g$; $[e,h] = h$, $[g,h] = k$
2	$\{P_t - P_z, P_y, A_x, A_y, K_z\}$	

TABLE X.3 - FIVE-DIMENSIONAL ALGEBRAS IN $\mathcal{P}$

(Beginning)

dim	Isomorphic Algebras	Non-isomorphic Algebras
		$[e,f] = \sigma f + h$; $[e,g] = g$; $[e,h] = h$; $[g,k] = h$
2		$\{P_t - P_z, P_y, A_x, A_y, \sigma K_z + P_x\}_{\sigma > 0}$
		$[e,f] = g$; $[e,g] = -f$; $[e,h] = -\alpha h$; $[e,k] = \alpha k$
4	$T(4) \circ F_{?}$ $\{P_t, P_x, P_y, P_z, J_z\}$	
		$[e,f] = f$; $[e,g] = -g$; $[e,h] = h$; $[f,k] = h$
3	$\{P_t, P_x, P_y, A_x, K_z\}$	$\{P_t, P_x, P_z, A_x, \sigma K_z + P_y\}_{\sigma > 0}$

TABLE X.3 - FIVE-DIMENSIONAL ALGEBRAS IN $\mathcal{P}$

(End)

Let us precise that in Table X.3 e , f , g , h , k represent the generators of any 5-dimensional algebra. In each case, we have written only the non-zero Lie products.

$\dim(\mathcal{L}_A + \mathcal{L}'_A)$	$\dim \mathcal{L}_A$	Algebra	Derivative Algebra D	dim D
6 5 4	0 1 2	Lorentz	Lorentz	6
3	3	$E(3)$: $\{J_x, J_y, J_z, P_x, P_y, P_z\}$	$\{\vec{J}, \vec{P}\}$	6
3	3	$E(2,1)$: $\{J_y, K_x, K_z, P_t, P_y, P_z\}$	$\{J_y, K_x, K_z, P_t, P_y, P_z\}$	6
3	3	$T^{s}(3) \circ E(2)$: $\{J_z, A_x, A_y, P_x, P_y, P_t - P_z\}$	$\{A_x, A_y, P_x, P_y, P_t - P_z\}$	5
3	3	$W(2) \circ H^{T(1)}$: $\{J_z \pm (P_t + P_z), A_x, A_y, P_x, P_y, P_t - P_z\}$	$\{A_x, A_y, P_x, P_y, P_t - P_z\}$	5
3	3	$T^{s}(3) \circ T(2)$: $\{K_z, A_x, A_y, P_x, P_y, P_t - P_z\}$	$\{A_x, A_y, P_t - P_z\}$	3
3	3	$T(3) \circ S_2(2)$: $\{J_z, A_x, A_y, P_x, P_y, P_t - P_z\}$	$\{A_x, A_y, P_x, P_y, P_t - P_z\}$	5
2	4	$E(2) \times E(1,1)$: $\{J_z, K_z, T(4)\}$	$T(4)$	4
2	4	$T(4) \square A(2)$: $\{A_x, A_y, T(4)\}$	$\{P_x, P_y, P_t - P_z\}$	3
2	4	$T(4) \circ S(1)$ $\{K_z, A_x, T(4)\}$	$\{P_x, P_z, P_t\}$	3

TABLE X.4 - SIX-DIMENSIONAL POINCARE SUBALGEBRAS

XI - THE SONS AND GRANDSONS OF THE SIX DIMENSIONAL CONNECTED SUBGROUPS OF P

In this section we give first the set of the sons and grandsons up to a Poincaré conjugation of the $T(4) \supset S(2)$ subgroup (Table XI.1). In the following tables we draw the same p.o.s. for the six-dimensional algebras appearing in Table XI.1. The p.o.s. corresponding to the three algebras L(3,1), E(3) and E(2,1) have been studied in ref. [1]. To build these p.o.s. use is made of the theorems on the maximal algebras given in section III (cf. ref [1]) and of the results of the previous sections of this paper.

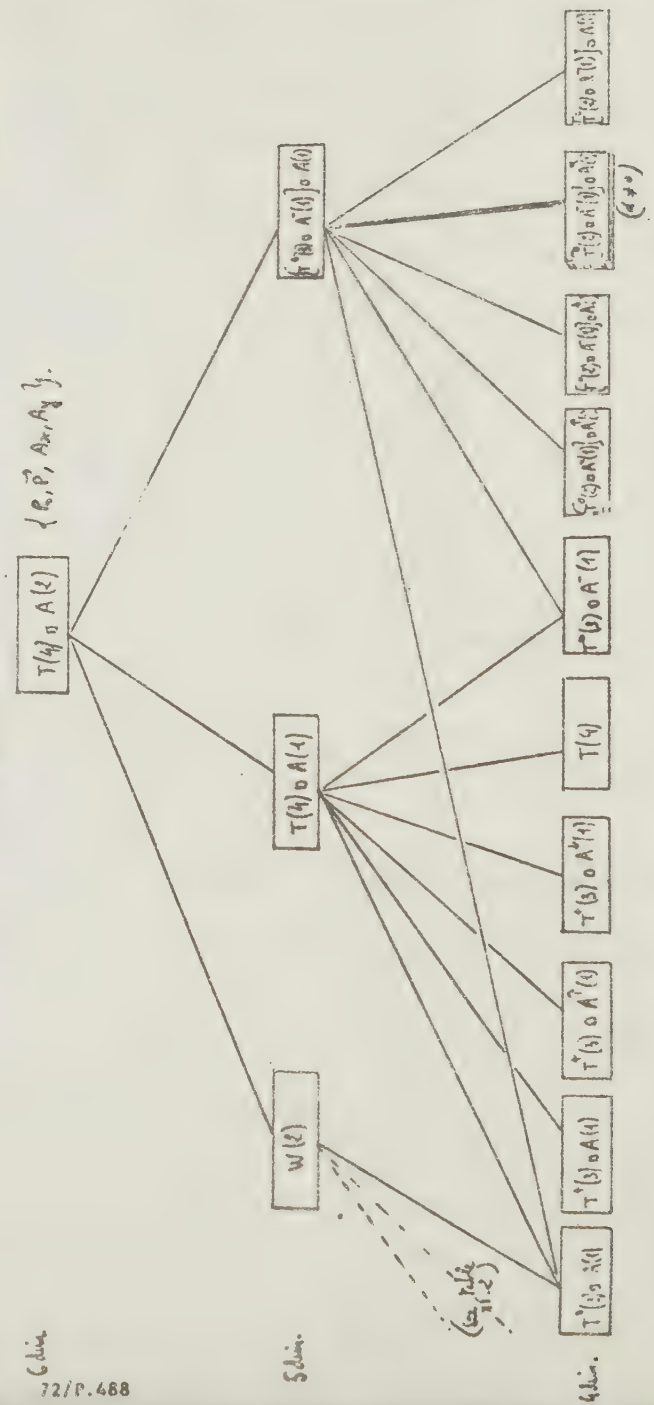

TABLE IX.4 - SONS AND GRANDSONS OF $T(4) \supset A(2)$

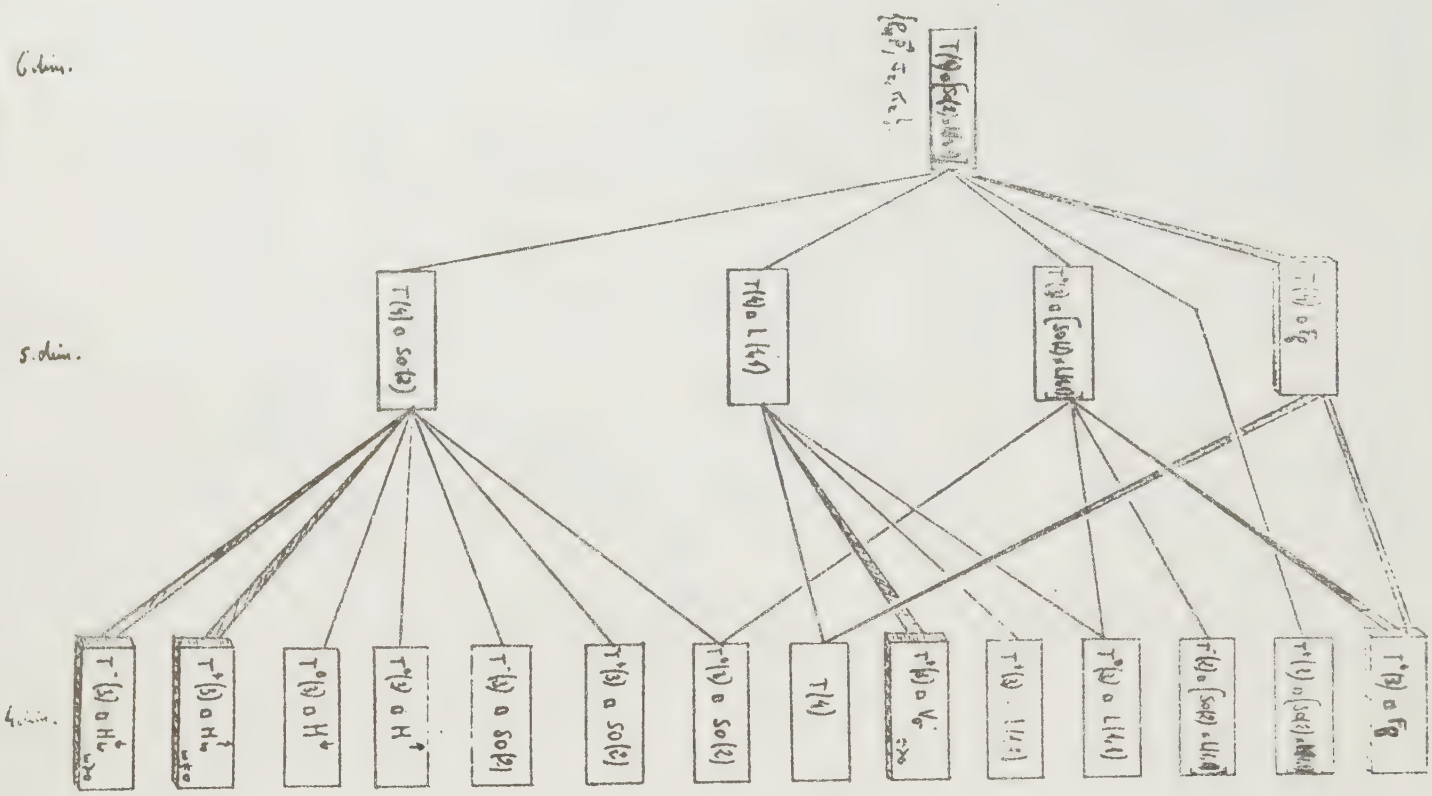

TABLE XI.5: - SONS AND GRANDSONS OF $T(4) \supset [S\ell(2) \times L(4,1)]$

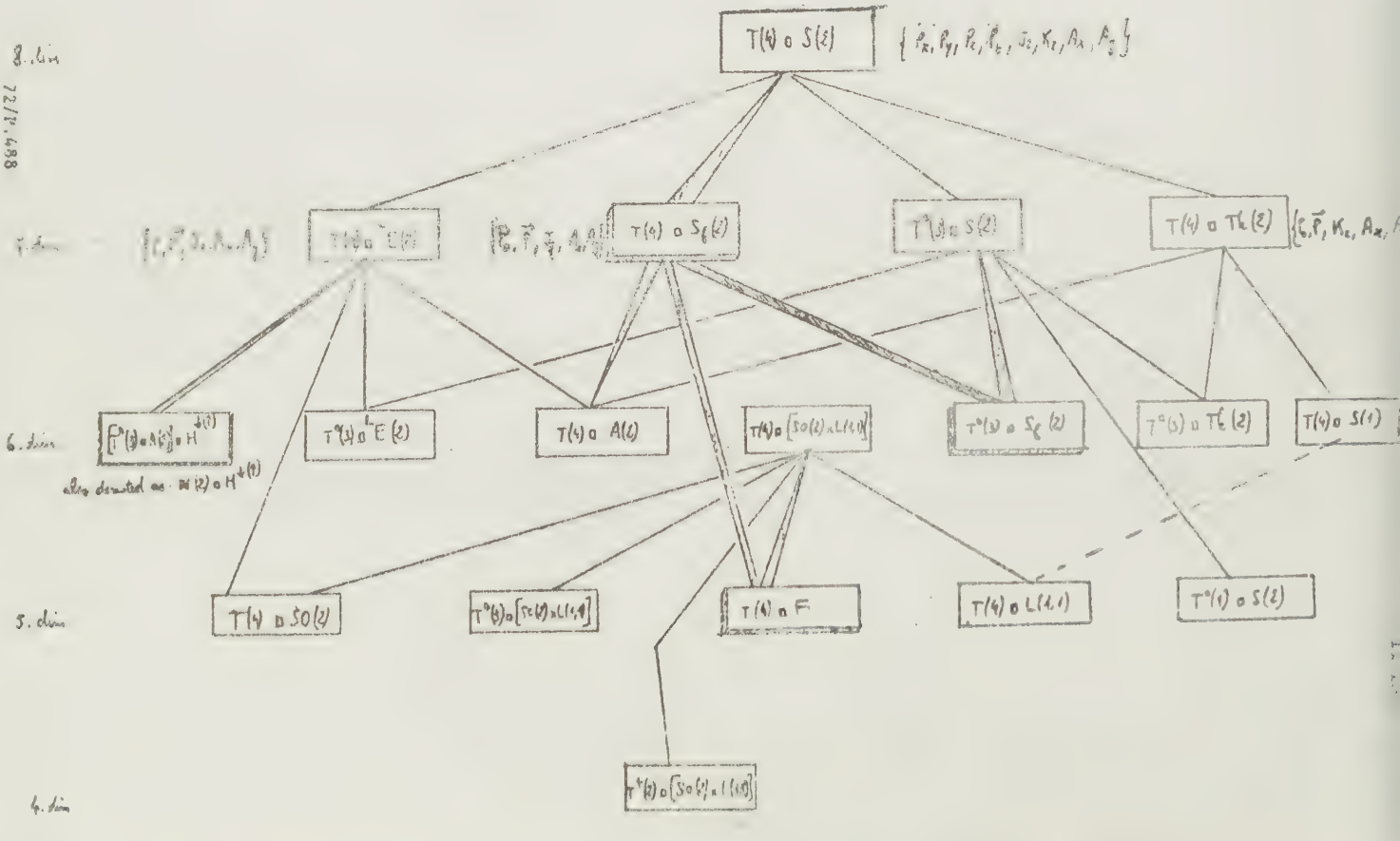

TABLE XI.1 - SONS AND GRANDSONS OF THE $T(4) \supset S(2)$

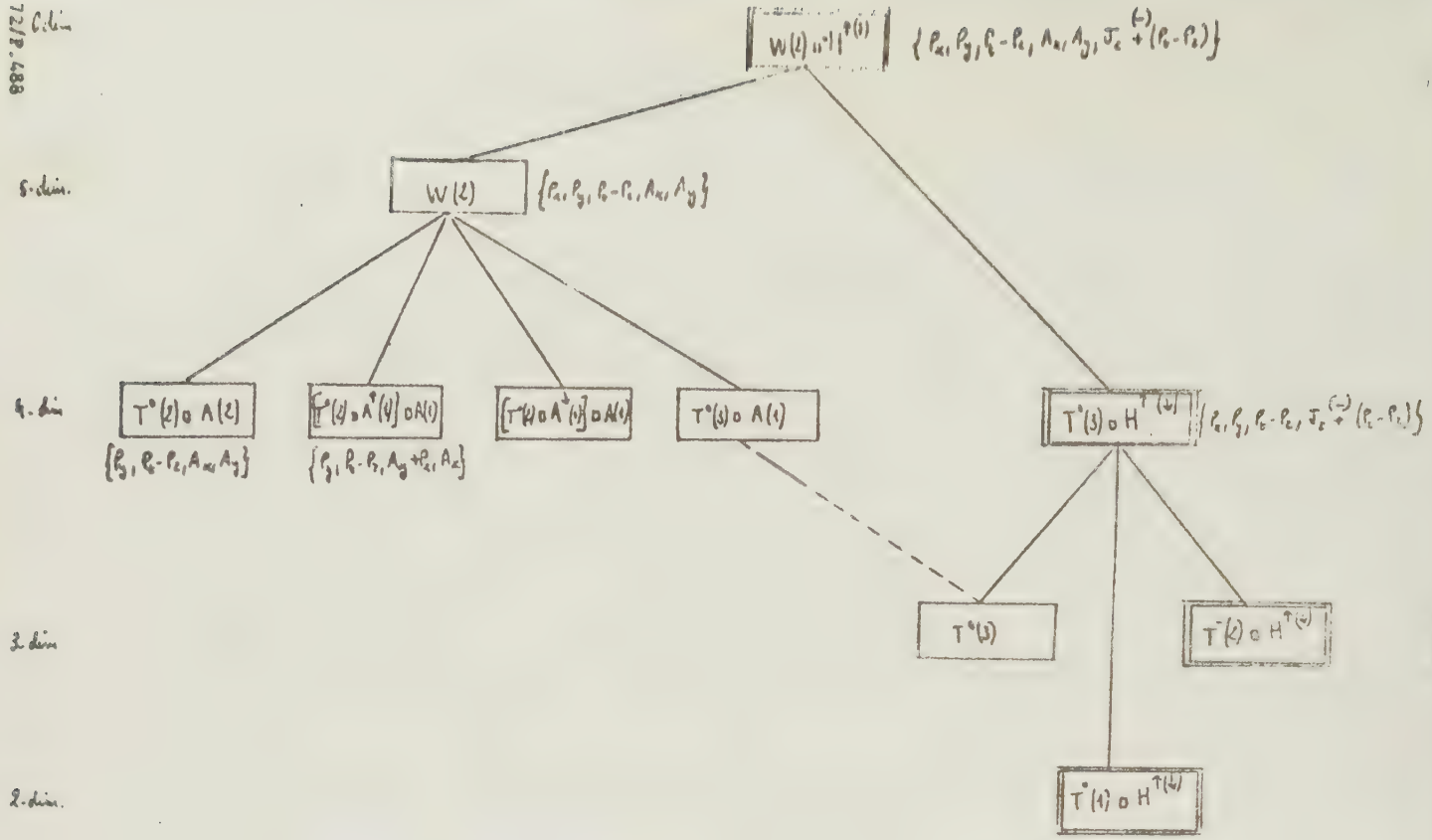

TABLE XI.2 - SONS AND GRANDSONS OF $W(2) \circ H^{T(2)}$

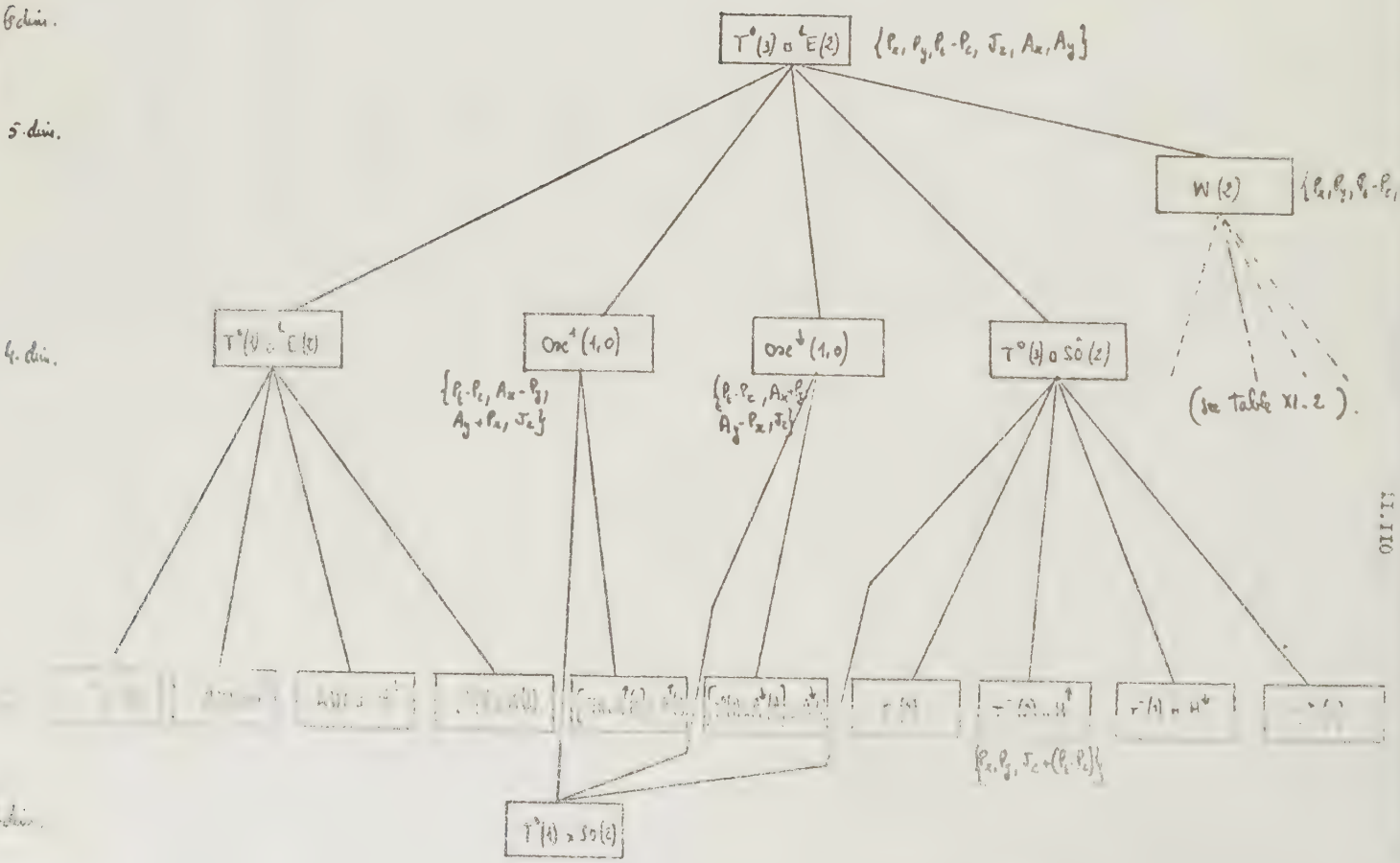

TABLE XI.3 - SONS AND GRANDSONS OF $T^{\circ}(3) \circ {}^{L}E(2)$

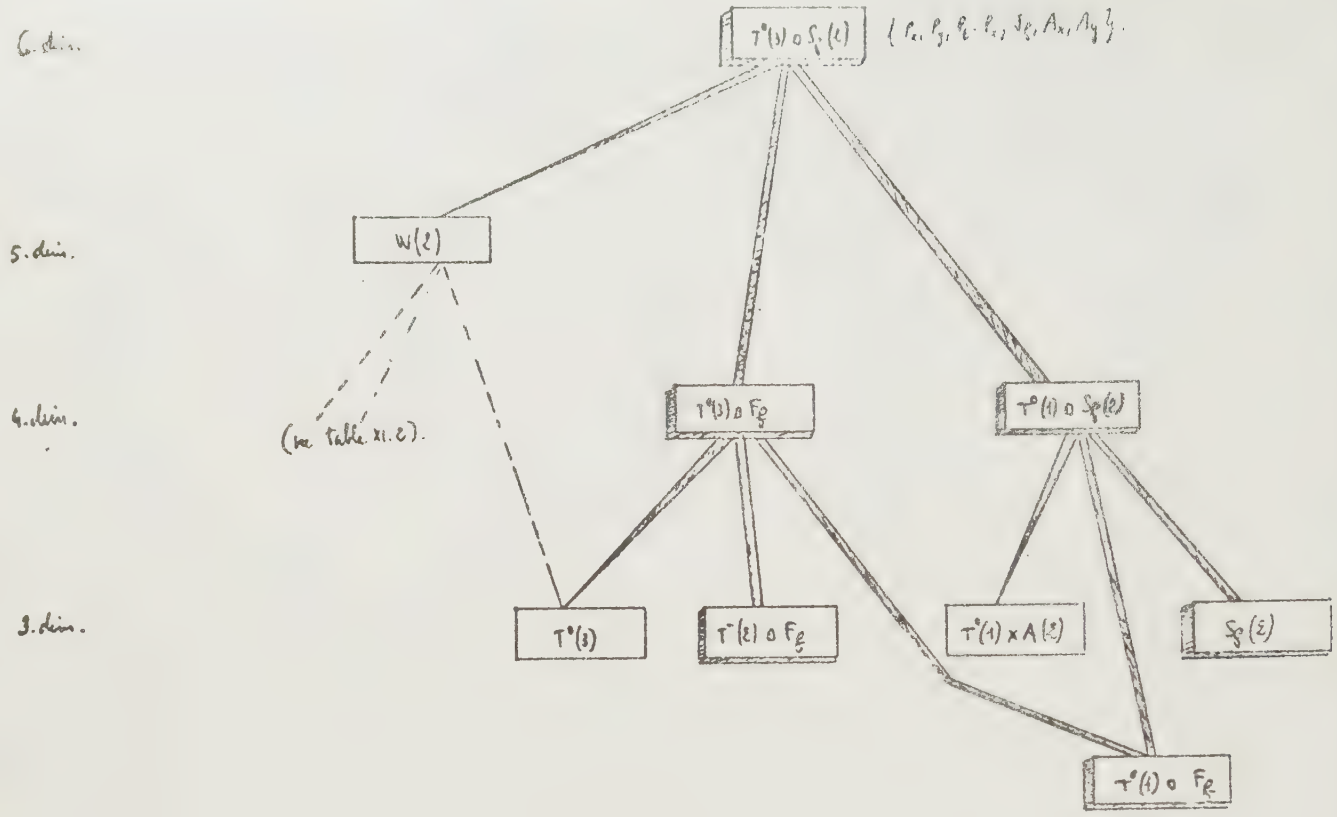

TABLE XI.6 - SONS AND GRANDSONS OF $T^{\circ}(3) \square S_{\rho}(2)$

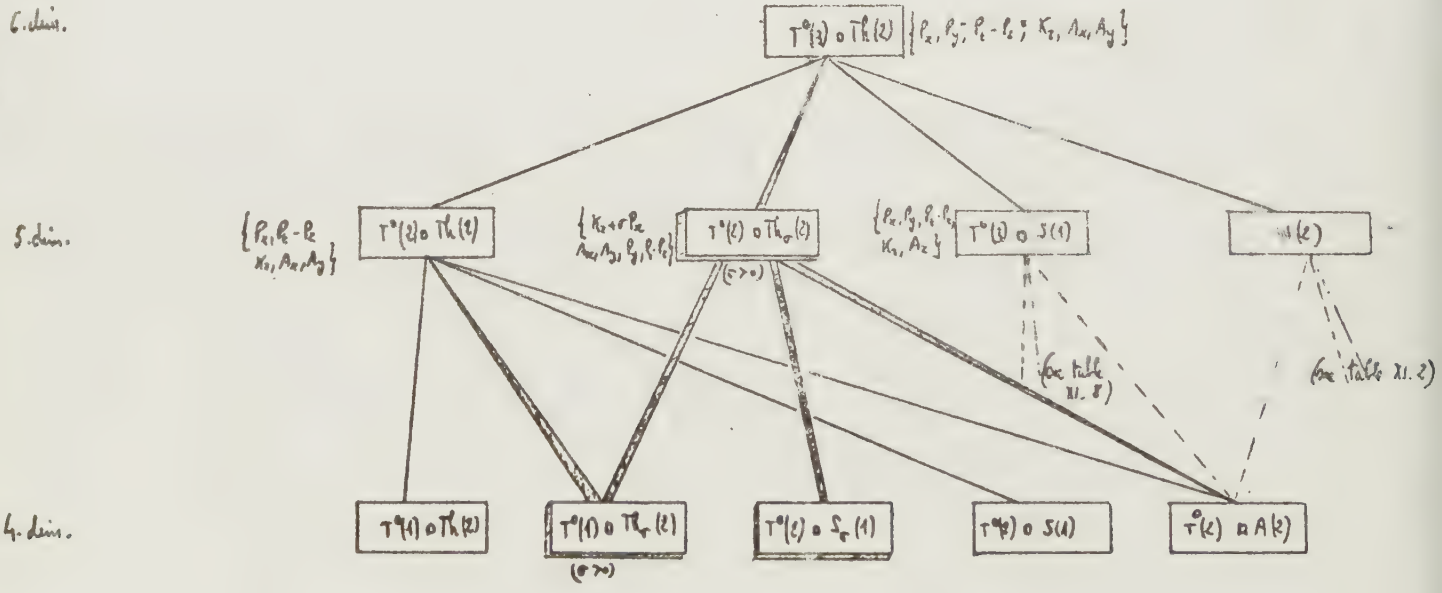

TABLE IX.7 - SONS AND GRANDSONS OF $T^{\circ}(3 \circ Tk(2)$

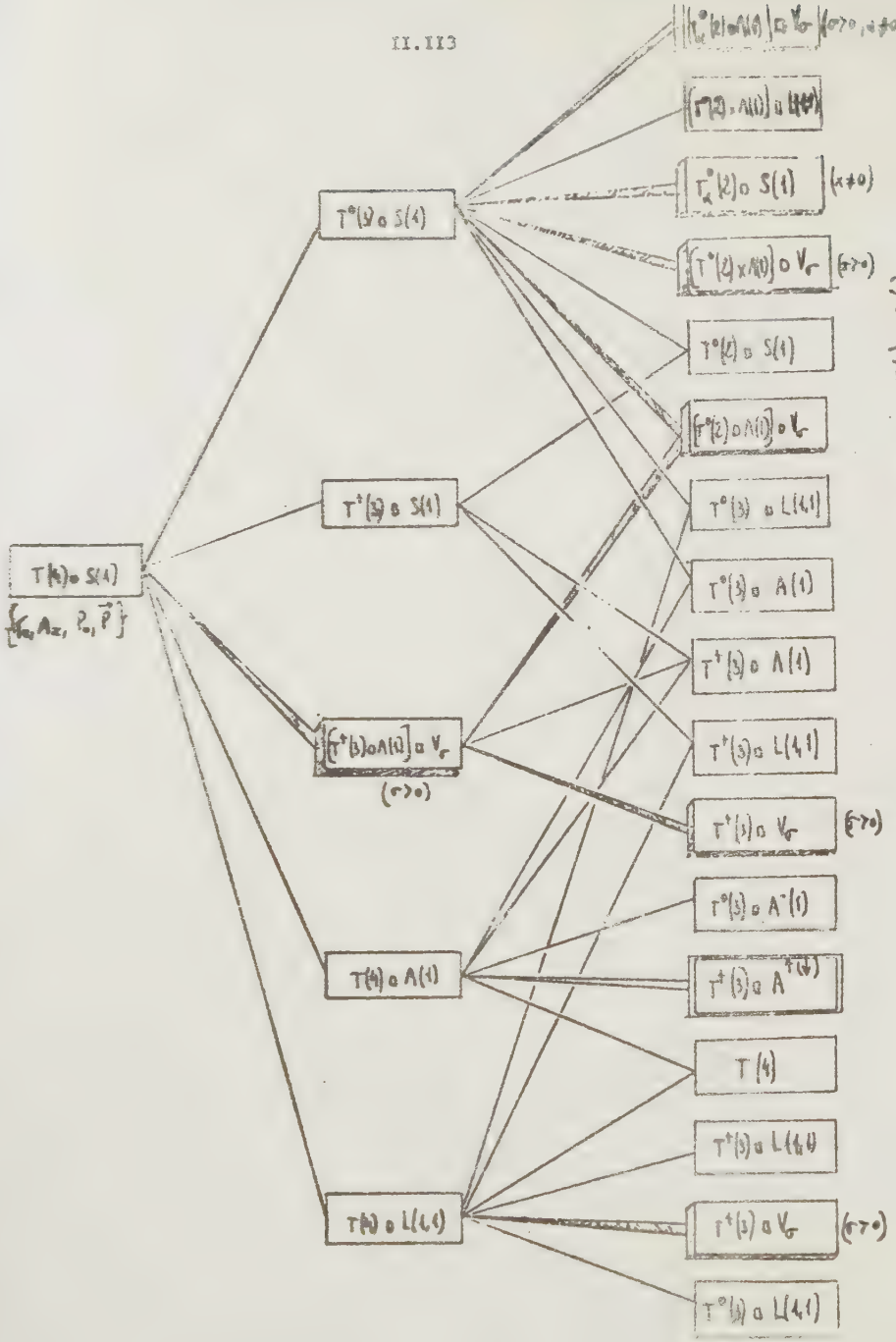

TABLE XI.8 - SONS AND GRANDSONS OF $T(4) \supset S(1)$

precise that $[T_2^\circ(2) \supset A(1)] \supset V_\sigma$ is generated by : $\sigma K_3 + P_x$, A_x , $P_x + \alpha P_y$, $P_0 - P_z$.

$T_4^\circ(3) \supset S(1)$ is generated by : K_z , A_x , $P_x + \alpha P_y$, $P_0 - P_z$.

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CHAPITRE 3.

ELECTROMAGNETIC FIELDS WITH SYMMETRY

Ph. COMBE *

and

P. SORBA **

ABSTRACT : In this paper, a study of the connected subgroups of the Poincaré group which can be seen as invariance groups of electromagnetic fields -they will be called c.e.P.s. - and of their corresponding invariant fields is investigated.
It appears that the dimension of such symmetry groups is less or equal to six. A method of determination of fields invariant under a Poincaré subgroup is proposed and applied to the characterization of all the electromagnetic fields invariant under each six and five dimensional c.e.P.s. .
Moreover, it can be shown that each connected Poincaré subgroup of dimension less or equal to four is either a c.e.P.s. or a subgroup of a c.e.P.s. .

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I - INTRODUCTION

The group theoretical definition of an elementary particle from continuous unitary irreducible ray representations of the Poincaré group $\mathcal{P}$ is well-known [1]. It is obvious that when an external field is present, the Poincaré symmetry is broken. However this external field may carry a subgroup of $\mathcal{P}$ as symmetry group, i.e. a group leaving invariant the field under consideration. In this last case informations about the "motion" of a particle in this field can be drawn from the study of the corresponding subgroup.

This sort of problem has been already studied for a charged system in a constant and uniform electromagnetic field [2] [3] and in the electromagnetic field of a circularly polarized plane wave [4] [5]. Interesting results have been drawn from such investigations, namely the characterization of the electric charge as a generator of the extended "kinematical" group of a constant and uniform electromagnetic field from which additivity of charge and the superselection rule for the electric charge have been derived. Moreover the analysis of the stability group of a circularly polarized plane wave has let exhibit the origin of the "effective mass" of an electron interacting with a laser beam, the laser beam being then considered as the circularly polarized plane wave.

Let us add that the knowledge of the symmetry group of a given electromagnetic field (E.M.F.) can be used to derive the most general form of the Lagrangian of a particle in such an external E.M.F. [6] [7].

Hence, it appears very promising to know among the connected subgroups of P those which have the property to be symmetry group for an E.M.F., and to exhibit the corresponding fields.

Let us just recall that an E.M.F. is a mapping from the Minkowsky space M into the space of skew-symmetric tensors of rank two, which verifies the Maxwell equations[1]:

$$\partial_\rho F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0 \qquad \text{(I.1)}$$

$$\partial_\mu F^{\mu\nu} + j^\nu = 0 \qquad \text{(I.2)}$$

where $\partial_\mu = \frac{\partial}{\partial x^\mu}$ $(\mu = 0, 1, 2, 3)$.

In this note, the field F is such that the components $F^{\mu\nu}$ are real functions defined and derivable on M or a common subspace of M.

This paper is organized as follows :
In section II, we define a connected electromagnetic Poincaré subgroup (c.e.P.s.) and a sub-c.e.P.s., and derive properties of such c.e.P.s. It is shown, in particular, that the dimension of a c.e.P.s. is less or equal to six.
In section III, we use the propositions of section II to eliminate, among the connected subgroups of P classified in [8] and [9], subgroups which are not c.e.P.s. or sub-c.e.P.s. Then we give a

(1) The used metric on the Minkowsky space M is defined by:
$g = \mathrm{diag}(+ - - -)$.

method to exhibit the E.M.F. corresponding to a given c.e.P.s.G.
It is interesting to note that the fields invariant under are found without the help of systems of partial differential equations, which could appear as a natural way for solving this type of problem.
A complete study of "high-dimensional" c.e.P.s., that is c.e.P.s. of dimension six and five, is given in section IV.
In tables (IV.1) and (IV.2) the explicit forms of fields invariant under the high-dimensional c.e.P.s. are summarized. From these results, it can be easily derived that every connected Poincaré subgroup of dimension less or equal to four is either a c.e.P.s. or a sub-c.e.P.s.
In Appendix A , the system of partial derivative equations to be satisfied by a field invariant under a one-dimensional Poincaré subgroup is deduced from the action of infinitesimal transformations. Indeed, the computation of E.M.F. invariant under certain low dimensional c.e.P.s. can be simplified by the consideration of such a system.
Finally, Appendix B is devoted to the study of symmetries of the electromagnetic plane waves.

II - DEFINITION AND PROPERTIES OF A CONNECTED ELECTROMAGNETIC POINCARE SUBGROUP

We denote (a,L) an element of the Poincaré group $\mathcal{P}$ acting on M, a being the translation part and L the Lorentz part. Under the action of (a,L), the field F is transformed into the field $^{(a,L)}F$ such that :

$$ {}^{(a,L)}F^{\mu\nu}(x) = L^{\mu}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}\left(L^{-1}(x-a)\right) . \qquad (II.1) $$

We call stabilizer in $\mathcal{P}$ of the field F and we denote $Stab_{\mathcal{P}}(F)$ the set of elements $(a,L) \in \mathcal{P}$ such that :

$$ {}^{(a,L)}F = F \qquad (II.2) $$

We shall restrict ourselves to the case of fields F whose components $F^{\mu\nu}$ are real functions defined and derivable on some domain of M.

Let us give the two definitions :

Definition (II.1)

A (connected) electromagnetic Poincaré subgroup ((c.)e.P.s.) $\mathcal{G}$ is a (connected) subgroup of $\mathcal{P}$ such that there exists at least one electromagnetic field F, $F \neq 0$ and $F^{\mu\nu}$ being real functions defined and derivable on M or some domain of M, and such that the (connected part of) $Stab_{\mathcal{P}}(F)$ is $\mathcal{G}$.

Definition (II-2)

A sub-(c.)e.P.s. is a (connected) subgroup $\mathcal{K}$ of $\mathcal{P}$ such that there exists $\mathcal{K}'$ (connected) subgroup of $\mathcal{P}$, $\mathcal{K}' \supset \mathcal{K}$ and $\mathcal{K}' \neq \mathcal{K}$ and such that every E.M.F. invariant under $\mathcal{K}$ is invariant under $\mathcal{K}'$.

Now some simple properties of c.e.P.s. can be stated, which will simplify their classification.

1) First, let us note :

Proposition II.1

Let $\mathcal{G}$ and $\mathcal{G}'$ be two conjugate c.e.P.s., which means that there exists $(\alpha,\Lambda) \in \mathcal{P}$ such that $\mathcal{G}' = (\alpha,\Lambda)\,\mathcal{G}\,(\alpha,\Lambda)^{-1}$. Then if F is invariant under $\mathcal{G}$, $^{(\alpha,\Lambda)}F$ defined by (II.1) is invariant under $\mathcal{G}'$.

This proposition is readily deduced from (II.1). It follows from this proposition that it is sufficient to look for the c.e.P.s. up to a conjugation of $\mathcal{P}$.

2) The study of orbits in M under the action of a c.e.P.s. $\mathcal{G}$ brings interesting remarks. Let F be invariant under $\mathcal{G}$, then Eqs (II.1) and (II.2) give

$$ L^{\mu}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x) = F^{\mu\nu}(Lx+a) $$

for any $x \in M$ and $(a,L) \in \mathcal{G}$.

Then if we know the value of F at x, Eq. (II.3) allows us to deduce the value of F at $x' = Lx+a$. In particular, if $F(x_0) = 0$, then F is zero at any point of

the orbit of x_0 under G. If F has no finite value at x_0, this persists all along the orbit of x_0.

Now, let x_0 be such that $F(x_0)$ has a finite non zero value, then with the value $F(x_0)$ of the field F at the point x_0 we associate a constant and uniform field F_{x_0} whose components are defined by $F_{x_0}^{\mu\nu} = F^{\mu\nu}(x_0)$.
This constant and uniform field is characterized by the two quantities,(2)

$$\Delta(F_{x_0}) = \frac{1}{2} F_{x_0}^{\mu\nu} F_{x_0\,\mu\nu}$$

$$\phi(F_{x_0}) = \frac{1}{2} F_{x_0}^{*\mu\nu} F_{x_0\,\mu\nu} \qquad \text{(II.4)}$$

Indeed, it is well-known [2] that,

i) If the quantity $\Delta + i\phi \neq 0$, there exists a Lorentz basis for which the vectors $\vec{E}$ ($E^i = F^{oi}$: electric part) and $\vec{B}$ ($B^i = \frac{1}{2}\varepsilon^{ijk} F_{jk}$: magnetic part) are parallel.

ii) If this quantity $\Delta + i\phi = 0$, the vectors $\vec{E}$ and $\vec{B}$ are orthogonal with same magnitude.

Then defining the type of F at x_0 by the type of the associated field F_{x_0} it is easy to verify,

Proposition II-2

For any field F invariant under a c.e.P.s. G the quantity

(2) Let us recall,

$$F^*_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$$\Delta(x) + i\phi(x) = \frac{1}{2} F_{\mu\nu}(x) F^{\mu\nu}(x) + \frac{i}{2} F^*_{\mu\nu}(x) F^{\mu\nu}(x)$$

is constant on the orbit of x under G.
Therefore the type of the field F is the same at two points x and x' belonging to the same orbit.

3) Let F be an electromagnetic field and suppose there exist $x_0 \in M$ and $(a,L) \in \mathcal{P}$ such that

$$F^{\mu\nu}(x) = L^\mu_\rho L^\nu_\sigma F^{\rho\sigma}(x_0) \qquad \text{(II.5)}$$

These equations are exactly the equations to be satisfied by the Lorentz transformations leaving the constant and uniform field $F_{x_0} = F(x_0)$ invariant. We know from [2] that the connected Lorentz part of the stabilizer of a constant and uniform electromagnetic field $F = (\vec{E}, \vec{B})$ is two-dimensional, its Lie algebra being generated by

$$G = \vec{B}.\vec{J} - \vec{E}.\vec{K} \quad ; \quad G^* = \vec{E}.\vec{J} + \vec{B}.\vec{K} \qquad \text{(II.6)}$$

if in the Lorentz algebra J_k, K_k $(k = x,y,z)$ denote the generators of the rotations and the boosts, respectively.
According to whether quantity $\Delta + i\phi$ vanishes or not, the stabilizer forms a group isomorphic to $A(2)$ generated by $A_x = K_x - J_y$ and $A_y = K_y + J_x$, or to $SO(2) \times L(1,1)$ generated by J_z and K_z.
These remarks lead (3) to the

(3) Let us note that a translational inner automorphism in $\mathcal{P}$ of the type $(x,\mathbb{1})(0,L)(-x,\mathbb{1}) = (x - Lx, L)$ leads on the Lie algebra $\mathcal{P}$ of $\mathcal{P}$ to the simple computation of the quantity

$$m = \ell + [l, \ell]$$

where $(x,\mathbb{1}) = e^{l}$, $(0,L) = e^{\ell}$ and $(x - Lx, L) = e^{m}$.

Proposition (II.3)

If a subgroup G of $\mathcal{P}$ stabilizes the field F, then at any point $x_0 \in M$ for which $F(x_0) \neq 0$ is finite, the set of transformations in G of the form $(x_0 - L x_0, L)$ is generated by a Lie algebra of dimension $\leq \ell$, conjugate to one of the two algebras $so(\ell) \times L(1,1)$ or $A(\ell)$, or to a subalgebra of these two algebras.

4) As a direct consequence of this simple result, one can state

Proposition (II.4)

Every c.e.P.s. is of dimension ≤ 6.

Indeed, from the identity $(x - L x_0, L) = (x_0, 1)(0, L)(x_0, 1)^{-1}$ we deduce that the c.e.P.s. G considered in proposition (II.3) is conjugate, by the translation $(-x_0, 1)$ to the subgroup G_{x_0}, the Lorentz part of which $\mathcal{L}_{x_0}$ is therefore of dimension $\leq \ell$. The Lie algebra $\mathcal{G}_{x_0}$ of G_{x_0} can be written, using the notation of [8]

$$\mathcal{G}_{x_0} = \mathcal{L}_{x_0} + \mathcal{M}_{x_0} + \mathcal{T}_{x_0} \qquad (II.6)$$

where $\mathcal{T}_{x_0}$ is the translational part and $\mathcal{M}_{x_0}$ a complementary subspace of $\mathcal{L}_{x_0} + \mathcal{T}_{x_0}$. Since the dimension of $\mathcal{M}_{x_0} + \mathcal{T}_{x_0}$ cannot be more than four, one deduces that $\mathcal{G}_{x_0}$ and then $\mathcal{G}$, the Lie algebra of G, is of dimension less than or equal to six.

III - THE C.E.P.S. AND THEIR CORRESPONDING INVARIANT FIELDS

In order to determine the c.e.P.s. and exhibit the corresponding E.M.F., let us note :

i) Proposition (II.1) insures us that the c.e.P.s. have only to be found in the classification of the connected subgroups of $\mathcal{P}$ up to a conjugation. Use is made of the tables given in [8] and [9].

ii) Propositions (II.3) and (II.4) allow us to show that the following subgroups cannot be c.e.P.s. or sub-c.e.P.s. (4).

a) all subgroups G such that $\dim G > 6$.

b) the six-dimensional subgroups :

$$G\{\mathcal{T}(4), K_z, A_x\}, \quad G\{\vec{P}, \vec{J}\}, \quad G\{\mathcal{T}^+(3), J_y, K_x, K_z\},$$
$$G\{\mathcal{T}^0(3), J_z, A_x, A_y\}, \quad G\{\vec{J}, \vec{K}\}. \qquad (III.1)$$

c) the five-dimensional subgroups :

$$G\{\mathcal{T}^+(3), A_x, \sigma K_z + P_y\}, \quad \sigma > 0.$$

(4) We shall denote by $G\{X, Y \ldots Z\}$ the connected group the Lie algebra of which is generated by $X, Y \ldots Z$. Let us add that the generators of the translational algebra are denoted by P_x, P_y, P_z, P_t. The 4-dimensional Lie algebra of the translations is denoted by $\mathcal{T}(4)$, while $\mathcal{T}^0(3)$, $\mathcal{T}^+(3)$, $\mathcal{T}^-(3)$ are the algebras generated by $\{P_x, P_y, P_t - P_z\}$, $\{P_x, P_z, P_t\}$, $\{P_x, P_y, P_z\}$ respectively.

Before proving this assertion, let us note that results of section IV allow us to insure that in (III.1) are contained all connected Poincaré subgroups which are not c.e.P.s. or sub-c.e.P.s.

Considering the above six-dimensional subgroups written in b), we can compute, by use of commutation relations of the Poincaré Lie algebra that at any point $x_0 \in M$, the Lorentz part $\mathcal{L}_{x_0}$ previously defined to establish proposition (II.3) is two-dimensional but non conjugate to $SO(2) \times L(1,1)$ or $A(2)$ for $G\{\mathcal{T}(4), K_z, A_x\}$, while it is three dimensional for the others. It follows from proposition (II.3) that these groups cannot be c.e.P.s. or sub-c.e.P.s.

In order to show that $G\{\mathcal{T}^+(3), A_x, \sigma K_z + \rho_y\}, \sigma \neq 0$, are not c.e.P.s, it is then sufficient to show that invariance under $G\{\mathcal{T}^+(3), A_x\}$ implies invariance under $G\{\mathcal{T}(4), A_x, A_y\}$. This last property can be shown as follows. Let F be an E.M.F. invariant under $G\{\mathcal{T}^+(3), A_x\}$. Then F depends only of the y-variable. Moreover invariance of F under elements L generated by A_x implies

$$F^{\mu}(Lx_0) = L^{\rho}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x_0)$$

for any x_0 belonging to the domain of definition of F. But considering the action[5] of L on M, we deduce that

(5) Using the basis $\{e_1, e_2, l, l'\}$ where $l = \frac{e_0 + e_3}{2}$ and $l' = e_0 - e_3$ we can compute

$$e^{\alpha A_x} = \begin{vmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \alpha & 0 & \frac{\alpha^2}{2} & 1 \end{vmatrix} \quad ; e^{\beta A_y} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \beta & \frac{\beta^2}{2} & 1 \end{vmatrix}$$

$F(Lx_0) = F(x_0)$. Then we can write :

$$F^{\mu}(x_0) = L^{\rho}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x_0)$$

Coming back to §3 of section II, it follows

$$F^{\mu\nu}(L'x_0) = L'^{\rho}_{\rho} L'^{\nu}_{\sigma} F^{\rho\sigma}(x_0)$$

with $L' = e^{\beta A_y}$, β real. As F vérifies also[5] : $F(L'x_0) = F(x_0)$, we have proved that F is invariant under any element $L' = e^{\beta A_y}$, β real. Now, by use of commutation relations in $\mathcal{L}$, we easily deduce that F invariant under $G\{\mathcal{T}^+(3), A_x\}$ is invariant under $G\{\mathcal{T}(4), A_x, A_y\}$.

Now, in order to study the subgroups of $\mathcal{L}$ which are not given in (III.1) and determine their corresponding invariant fields, if any, we use the following method :

i) We look at the orbits in M under G.

ii) From section (II.3), on every orbit one is able to see if there exists a finite non zero field and eventually to characterize its type. Hence we get informations about the domain of M in which such a field is defined.

iii) Once the value of the field is given on a representative point of each orbit, using Eq. (II.3) we derive the value of the field at each point of the orbit.

iv) Now some continuity condition could be imposed, for instance, the continuity on the domain of M where the field has finite values.

v) We impose the Maxwell equations (I.1) in order to select the E.M.F., the equations (I.2) being considered as a definition of the 4-current j^{μ}.

Two remarks can then be added.

The first one is relative to the case where we should like to consider the presence of magnetic charges. Then Eq. (I.1) must be changed into : $\partial_{\rho} F^{*\rho\nu} = j^{*\nu}$

and considered as a definition of the 4-magnetic current
The second one concerns the physical realization of the fields. It may happen that fields satisfying the first Maxwell equation (I.1) lead by (I.2) to currents which cannot be realizable. An example is given by fields corresponding to charged particle moving with light velocity (see table IV.2).

We illustrate the method on some examples.

1) Let us consider the group $G = T^4(4) \times So(3)$ generated by $\{P_t, J_x, J_y, J_z\}$. The orbits in M are of the following type

i) the t-axis $(x = y = z = 0)$.

ii) the direct product $R \times S_{0,r}$ of the t-axis by the sphere the equation of which being

$$x^2 + y^2 + z^2 = r^2 \qquad r > 0 \tag{III.2}$$

First let us note that the invariant fields do not depend on the t-variable. Moreover proposition (II.3) insures that there does not exist any finite non zero field on the orbit i). In the case ii) we choose as representative point of the orbits[6]

$$x_0 = (0, 0, 0, r) \qquad r > 0$$

The set of transformations in G of the form $(x_0 - L x_0, L)$ is generated by J_z. So the constant and uniform field $F_{\mu\nu} = F(x_0)$ is of the $So(2) \times L(1,1)$ type generated by J_z and K_z.

(6) We choose here the basis $\{e_0, e_1, e_2, e_3\}$.

Then from Eq. (II.6) we can write

$$\left.\begin{aligned} &E_x(0,0,r) = E_y(0,0,r) = B_x(0,0,r) = B_y(0,0,r) = 0 \\ &E_z(0,0,r) = f(r) \\ &B_z(0,0,r) = g(r) \end{aligned}\right\} \tag{III.3}$$

By action of the element $e^{\alpha J_x} e^{\theta J_y}$ of G

$$e^{\alpha J_x} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{vmatrix} \qquad e^{\theta J_y} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta \end{vmatrix} \tag{III.4}$$

We obtain

$$\left.\begin{aligned} &E_x(r\sin\theta, -r\cos\theta\sin\alpha, r\cos\theta\cos\alpha) = \sin\theta\, f(r) \\ &E_y(r\sin\theta, -r\cos\theta\sin\alpha, r\cos\theta\cos\alpha) = -\cos\theta\sin\alpha\, f(r) \\ &E_z(r\sin\theta, -r\cos\theta\sin\alpha, r\cos\theta\cos\alpha) = \cos\theta\cos\alpha\, f(r) \end{aligned}\right\} \tag{III.5}$$

The same type of results are obtained for $\vec{B}$, and putting

$$x = r\sin\theta \;,\; y = -r\cos\theta\sin\alpha \;,\; z = r\cos\theta\cos\alpha$$

we get

$$\left.\begin{aligned} &E_x(x,y,z) = \frac{x}{r} f(r) \qquad B_x(x,y,z) = \frac{x}{r} g(r) \\ &E_y(x,y,z) = \frac{y}{r} f(r) \qquad B_y(x,y,z) = \frac{y}{r} g(r) \\ &E_z(x,y,z) = \frac{z}{r} f(r) \qquad B_z(x,y,z) = \frac{z}{r} g(r) \end{aligned}\right\} \tag{III.6}$$

Now we assume f and g to be continuous and derivable for $r>0$. The first Maxwell equations (I.I) give

$$g(r) = \frac{A}{r^2} \qquad A \text{ real.} \tag{III.7}$$

Hence the fields invariant under $T^4(1) \times SO(3)$ are of the following form

$$\left.\begin{aligned} \vec{B}(x) &= A\,\frac{\vec{x}}{r^2} \\ \vec{E}(x) &= \vec{x}\, h(r) \end{aligned}\right\} \qquad r = x^2+y^2+z^2. \tag{III.8}$$

In our example the field is defined in the complement of the t-axis

II) As a second example let us study the case of the group $G' = T^4(1) \times SO(2,1)$ generated by $\{P_z, J_z, K_x, K_y\}$. In the previous example, The Lorentz part isomorphic to $SO(3)$ were compact while in this case the Lorentz part is isomorphic to $SO(2,1)$ and therefore non-compact.

The orbits in M under G' can be classified as follows

i) The z-axis $(t = x = y = 0)$

ii) The direct products $R \times C_{z\pm}$ of the z-axis by the half-cones

$$t^2 - x^2 - y^2 = 0 \qquad \begin{matrix} \alpha)\ t>0 \\ \beta)\ t<0 \end{matrix} \tag{III.9}$$

iii) The direct products of the z-axis by the hyperboloids, or sheets of hyperboloids, defined by

$$t^2 - x^2 - y^2 = k \qquad \begin{matrix} \alpha)\ k>0,\ t>0 \\ \beta)\ k>0,\ t<0 \\ \gamma)\ k<0 \end{matrix} \tag{III.10}$$

There exists no finite, non-zero fields corresponding to the case i). In the case ii) a simple computation leads to the solutions

$$\left.\begin{aligned} E_x &= E_y = B_z = 0 \\ B_z &= t\,a_\varepsilon \\ B_x &= y\,a_\varepsilon \\ B_y &= -x\,a_\varepsilon \end{aligned}\right\} \tag{III.11}$$

where $t = \varepsilon\sqrt{x^2+y^2}$, $\varepsilon = \pm 1$ according to the case ii, α) or ii, β), a_ε being constants.

In the case iii, α, β) we obtain

$$\left.\begin{aligned} E_x &= A_\varepsilon \frac{y}{k^{1/2}} & B_x &= y\,f(k) \\ E_y &= A_\varepsilon \frac{x}{k^{1/2}} & B_y &= -x\,f(k) \\ E_z &= t\,f(k) & B_z &= A_\varepsilon \frac{t}{k^{1/2}} \end{aligned}\right\} \tag{III.12}$$

where $t = \varepsilon\sqrt{k+x^2+y^2}$ with $k>0$, $\varepsilon = \pm 1$ according to the case α or β, A_ε being constants.

We assume f to be continuous and derivable in $k = t^2-x^2-y^2>0$. Then we have a field defined on the strata corresponding to

$k>0$, $t>0$ or $t<0$. Finally in the case iii, γ) the invariant electromagnetic field is given by

$$\left.\begin{aligned} E_x &= -\beta\frac{y}{k^{3/2}} & B_x &= \gamma\, g(k)\\ E_y &= \beta\frac{x}{k^{3/2}} & B_y &= -\alpha\, g(k)\\ E_z &= t\, g(k) & B_z &= \beta\frac{t}{k^{3/2}} \end{aligned}\right\} \qquad \text{(III.13)}$$

where β is a constant and g a continuous and derivable function on the strata $k = t^2-x^2-y^2<0$.

Combining these different fields, we can easily see that there exists a field defined on the complement part of the z-axis, it is given by

$$\left.\begin{aligned} E_x &= E_y = B_z = 0\\ E_z &= t\, h\left(t^2-x^2-y^2\right)\\ B_x &= \gamma\, h\left(t^2-x^2-y^2\right)\\ B_y &= -\alpha\, h\left(t^2-x^2-y^2\right) \end{aligned}\right\} \qquad \text{(III.14)}$$

where h is a continuous and derivable function in the $k = t^2-x^2-y^2$ variable.

IV - CLASSIFICATION OF THE HIGH DIMENSIONAL C.E.P.S. WITH THEIR CORRESPONDING INVARIANT FIELDS.

In this section, a complete study of high dimensional, that is six and five dimensional c.e.P.s. with the classification of their corresponding invariant fields, is achieved. It appears that among the five dimensional connected Poincaré subgroups, several of them are only sub-c.e.P.s. All these results are gathered in tables (IV.1) and (IV.2).

An interesting consequence is that (III.1) gives all the connected subgroups of $\mathcal{P}$ which are not c.e.P.s. or sub-c.e.P.s.

1) Six dimensional c.e.P.s.

Following table (X.4) of Ref [9] and (III.1) , we are interested only by the following groups :

$$G\{\mathcal{T}(4), T_z, K_z\}\ ,\ G\{\mathcal{T}(4), A_x, A_y\}\ ,\ G\{\mathcal{T}^0(3), T_z+\varepsilon(\ell_1+\ell_2), A_x, A_y\}\ \varepsilon=\pm 1$$
$$G\{\mathcal{T}^0(3), K_z, A_x, A_y\}\ ,\ G\{\mathcal{T}^0(3), T_\xi, A_x, A_y\}\ \text{with } T_\xi = T_z\cos\tfrac{\xi}{\ell} - K_z\sin\tfrac{\xi}{\ell}$$
$$\text{and } \sin\xi \neq 0 .$$

The two first groups are the symmetry groups of constant and uniform electromagnetic field. Using the relations (II.6) , one can readily verify the corresponding solutions given in table (IV.1).

We recognize in the groups $G\{\mathcal{T}^0(3), J_z + \varepsilon(P_0 + P_3), A_x, A_y\}$ $\varepsilon = \pm 1$, the symmetry groups of circularly polarized plane waves. Using the method developed in section III, we can compute that there do not exist other solutions.

Now, let us consider more carefully the group $G\{\mathcal{T}^0(3), K_z, A_x, A_y\}$. Let us suppose F to be an E.M.F. invariant under this group. Invariance under $\mathcal{T}^0(3)$ implies that F depends only of the $t-z$ variable. Moreover, Proposition II.3 allows to deduce that F cannot be non-zero and defined at $t-z=0$. The same proposition insures that at any point x_0 such that $t-z \neq 0$ and where F is defined and not zero, $F(x_0)$ is of the type $A(2)$, the stabilizer of $F(x_0)$ being generated by A_x and A_y. It is easy to compute, using (II.6) that at such points the field verifies

$$\left.\begin{aligned} E_x(x_0) &= B_y(x_0) \\ E_y(x_0) &= -B_x(x_0) \\ E_z(x_0) &= B_z(x_0) = 0 \end{aligned}\right\} \quad \text{(IV.1)}$$

Now consider the invariance of F under boosts along the z-axis. It is well-known that the 4x4 matrix L corresponding to $e^{\theta K_z}$, θ real, acting on a four-vector $(t, x, y, z) \in M$ can be written

$$e^{\theta K_z} = \begin{vmatrix} ch\theta & 0 & 0 & sh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ sh\theta & 0 & 0 & ch\theta \end{vmatrix} \quad \text{(IV.2)}$$

Eq. (II.3) furnishes then the relations :

$$\left.\begin{aligned} E_x\left(e^{-\theta}(t-z)\right) &= e^{\theta} E_x(t-z) \\ E_y\left(e^{-\theta}(t-z)\right) &= e^{\theta} E_y(t-z) \end{aligned}\right\} \quad \text{(IV.3)}$$

Hence, we can deduce the only fields invariant under $G\{\mathcal{T}^0(3), K_z, A_x, A_y\}$:

$$\left.\begin{aligned} &\cdot\ E_x(t-z) = B_y(t-z) = \frac{A}{t-z} \\ &\quad E_y(t-z) = -B_x(t-z) = \frac{B}{t-z} \\ &\quad E_z = B_z = 0 \qquad A \text{ and } B \text{ real constants.} \end{aligned}\right\} \quad \text{(IV.4)}$$

these functions been defined, continue and derivable in the set $t-z>0$ or $t-z<0$.

Note these solutions verify the Maxwell equations (I.1) and correspond to a vanishing current $(j^\nu = 0)$; see Appendix B .

The same type of computation can be used to show that the groups $G\{\mathcal{T}^0(3), J_z, A_x, A_y\}$ are c.e.P.s. and furnish the solutions given in table (IV.1) .

2) Five-dimensional c.e.P.s.

It is easy to remark, using results of section (II.3) that a five dimensional subgroup of $\mathcal{P}$ such that its translatio-

nal part is four dimensional, is a sub-c.e.P.s.
Moreover we can prove that if a field F is invariant under $G\{\mathcal{T}^*(3), A_x\}$, it is invariant under $G\{\mathcal{T}^*(3), A_x, A_y\}$. Indeed invariance under $G\{\mathcal{T}^*(3), A_x\}$ implies that F depends only of the t-z variable. Eqs. (II.3) give at any point $x_0 = (0,0,\gamma,0)$ in the basis $\{e_1, e_2, \ell, \ell'\}$

$$F^{\mu\nu}(Lx_0) = F^{\mu\nu}(x_0) = L^{\mu}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x_0) \qquad (IV.5)$$

with $L = e^{\alpha A_x}$, α real.
It follows from section II.3 that F verify also the relation

$$F^{\mu\nu}(L'x_0) = F^{\mu\nu}(x_0) = L'^{\mu}_{\rho} L'^{\nu}_{\sigma} F^{\rho\sigma}(x_0) \qquad (IV.6)$$

with $L' = e^{\beta A_y}$.

Using the same type of arguments we can prove that if the field F is invariant under $G\{P_1, P_0 - P_2, A_x, A_y\}$ it is invariant under $G\{\mathcal{T}^*(3), A_x, A_y\}$.

These simple properties lead us to restrict our attention in table (X.3) of Ref. 9 in which we have eliminated the groups $G\{\mathcal{T}^*(3), \sigma K_z + P_y, A_x\}$, $\sigma > 0$, owing to (III.1), to the three groups $G\{\mathcal{T}^*(3), A_x, A_y\}$, $G\{\mathcal{T}^*(3), J_z, K_z\}$ and $G\{P_0 - P_2, J_z, K_z, A_x, A_y\}$.
We use the technics already discussed to determine for each of these three groups their corresponding E.M.F. The results are summarized in table (IV.2).

3) c.e.P.s. and sub-c.e.P.s. of dimension less or equal less or equal to four

Proposition IV.1

Every connected Poincaré subgroup of dimension less or equal to four is either a c.e.P.s. or a sub-c.e.P.s.

Using once more the classification of connected Poincaré subgroups given in [8] and [9], it is easy to remark that the only subgroups of dimension less or equal to four which are not subgroups of six and five dimensional c.e.P.s. are the following : $G\{P_0, J_x, J_y, J_z\}$, $G\{P_z, J_z, K_x, K_y\}$, $G\{J_x, J_y, J_z\}$, $G\{J_z, K_x, K_y\}$. But it has been shown in section III that the two groups $G\{P_0, J_x, J_y, J_z\}$ and $G\{P_z, J_z, K_x, K_y\}$ are c.e.P.s. Moreover, we remark that the two groups $G\{J_x, J_y, J_z\}$ and $G\{J_z, K_x, K_y\}$ are subgroups of $G\{P_0, J_x, J_y, J_z\}$ and $G\{P_z, J_z, K_x, K_y\}$ respectively.

SIX DIMENSIONAL C.E.P.S.

$\dim \mathcal{T}_A$ [8]	c. e. P. s.	E. M. F.
4	$G\{\mathcal{E}(4), J_z, K_z\}$ (7)	$E_x = E_y = B_x = B_y = 0$ $E_z = \alpha$ $B_z = \beta$ α, β real
	$G\{\mathcal{E}(4), A_x, A_y\}$ (7)	$E_x = B_y = \alpha$ $E_y = -B_x = \beta$ $E_z = B_z = 0$ α, β real
3	$G\{\mathcal{E}^0(3), J_z + \varepsilon(P_1 + P_2), A_x, A_y\}$ $\varepsilon = \pm 1$ (8)	$E_x(t-z) = B_y(t-z) = A\cos\left(\frac{t-z}{2} + \theta\right)$ $E_y(t-z) = -B_x(t-z) = \varepsilon A \sin\left(\frac{t-z}{2} + \theta\right)$ $E_z = B_z = 0$ A, θ real (note that θ can be chosen $= 0$)
	$G\{\mathcal{E}^0(3), K_z, A_x, A_y\}$	$E_x(t-z) = B_y(t-z) = \frac{A}{t-z}$ $E_y(t-z) = -B_x(t-z) = \frac{B}{t-z}$ $E_z = B_z = 0$ A, B real $\vec{E}$ and $\vec{B}$ defined, continuous and derivable in the set $t-z<0$ or $t-z>0$
	$G\{\mathcal{E}^0(3), J_\ell, A_x, A_y\}$ $J_\ell = J_z \cos\frac{\ell}{\xi} - K_z \sin\frac{\ell}{\xi}$ $\sin\ell \neq 0$	$E_x(t-z) = B_y(t-z) = \frac{A}{t-z}\sin\left(\varphi \log|t-z| + \theta\right)$ $E_y(t-z) = -B_x(t-z) = -\frac{A}{t-z}\cos\left(\varphi \log|t-z| + \theta\right)$ $E_z = B_z = 0$ $\quad \varphi = 1/\sin\ell$ A, θ real (note that θ can be chosen $= 0$).

TABLE (IV.I)

(7) The continuous unitary irreducible ray representations of these groups are given in [2] and [6].

(8) $\mathcal{T}_A$ is the translational part of the Lie algebra $\mathcal{A}$ of the group G.

FIVE DIMENSIONAL C.E.P.S.

$\dim \mathcal{T}_A$	c.e.P.s. (or sub-c.e.P.s.)	E.M.F.
4	$G\{\mathcal{E}(4), J_z\}$ $G\{\mathcal{E}(4), K_z\}$ $G\{\mathcal{E}(4), J_\ell\}$	sub-c.e.P.s. of $G\{\mathcal{E}(4), J_z, K_z\}$ see table (III.1)
	$G\{\mathcal{E}(4), A_x\}$	sub-c.e.P.s of $G\{\mathcal{E}(4), A_x, A_y\}$
3	$G\{\mathcal{E}^0(3), A_x, A_y\}$	$E_x(t-z) = B_y(t-z)$ $E_y(t-z) = -B_x(t-z)$ $E_z = B_z = 0$
	$G\{\mathcal{E}^0(3), J_z, K_z\}$ (9)	$E_x = E_y = B_x = B_y = 0$ $E_z = A$ for $t-z>0$ $\quad E_z = A'$ for $t-z<0$ $B_z = C$
	$G\{\mathcal{E}^0(3), K_z, A_x\}$	sub-c.e.P.s. of $G\{\mathcal{E}^0(3), K_z, A_x, A_y\}$
	$G\{\mathcal{E}^0(3), A_x + P_0 + P_z, A_y\}$	sub-c.e.P.s. of $G\{\mathcal{E}(4), A_x, A_y\}$
2	$G\{P_y, P_t - P_z, K_z, A_x, A_y\}$ $G\{P_y, P_t - P_z, \sigma K_z + P_x, A_x, A_y\}$ $\sigma > 0$	sub-c.e.P.s. of $G\{\mathcal{E}^0(3), K_z, A_x, A_y\}$
1	$G\{P_t - P_z, J_z, K_z, A_x, A_y\}$ (9)	$E_x(x,y,t-z) = B_y(x,y,t-z) = \frac{A_x}{t-z}$ $E_y(x,y,t-z) = -B_x(x,y,t-z) = \frac{A_y}{t-z}$ $E_z(x,y,t-z) = -2A$ A real $B_z = 0$ defined $t-z>0$ or $t-z<0$

TABLE (IV.2)

(9) Note that the currents relative to these fields correspond to charged particles moving with light velocity.

V - CONCLUSION

In this note, the E.M.F. under consideration are such that their components $F_{\mu\nu}$ are real functions defined and derivable on some domain of M. Then, we have been able to select all connected Poincaré subgroups which leave invariant such fields : the c.e.P.s. and sub-c.e.P.s. Moreover for a given invariance group, we have shown how to determine all fields invariant under this group. Then we have applied this method for all six and five dimensional c.e.P.s. (high-dimensional c.e.P.s.).

An interesting result is the existence of several six dimensional c.e.P.s. other than the already well-known invariance groups of constant and uniform E.M.F. and circularly polarized plane waves. Let us note that the E.M.F. corresponding to these invariance groups are plane waves (see Appendix B).

The previous study can be continued by use of the same methods in order to classify the E.M.F. invariant under c.e.P.s. of dimension less or equal to four. Let us recall that two interesting cases have been investigated in section III.

Now it would be interesting to consider the $F_{\mu\nu}$ as tempered distributions. It is obvious that our methods of selection of invariance Poincaré subgroups and determination of the corresponding solutions are no more valid, because of the use of local properties of fields in our propositions. Moreover, it can be noted that in some particular cases (invariance under a great number of translations, for instance) the system of partial differential equations given in Appendix A can be used without difficulty to determine the fields solutions.

- APPENDIX A -

INFINITESIMAL TRANSFORMATIONS

Let F be an E.M.F. invariant under the one parameter Lie group, an element of which is $(a,L) \in \mathcal{L}$. By use of Eq.(II.3)

$$F^{\mu\nu}(Lx+a) = L^{\mu}_{\rho} L^{\nu}_{\sigma} F^{\rho\sigma}(x)$$

we can deduce, considering an infinitesimal transformation $L = 1 + \omega$

$$\omega \wedge F(x) + (x, \omega . \nabla) F(x) - (a . \nabla) F(x) = 0 \qquad (A.1)$$

where

$$(\omega \wedge F)_{\mu\nu} = \omega_{\mu}{}^{\lambda} F_{\lambda\nu} - \omega_{\nu}{}^{\lambda} F_{\lambda\mu}$$

and $\nabla = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial \vec{x}} \right)$ with $x = (x^0, \vec{x})$

By setting

$$\begin{cases} F = (\vec{E}, \vec{B}) & \text{with } E^i = F^{0i} \text{ and } B^i = \frac{1}{2} \varepsilon^{ijk} F_{jk} \\ \omega = (\vec{\varphi}, \vec{\theta}) & \text{with } \varphi^i = \omega^{0i} \text{ and } \theta^i = \frac{1}{2} \varepsilon^{ijk} \omega_{jk} \end{cases} \qquad (A.2)$$

Eq. (A.1) can be written

$$\left.\begin{aligned}
&\vec{\varphi}\times\vec{B} + \vec{\theta}\times\vec{E} + (\vec{x}.\vec{\varphi})\frac{\partial\vec{E}}{\partial t} + \left(t\vec{\varphi} + \vec{x}\times\vec{\theta}\right)\frac{\partial}{\partial\vec{x}}\vec{E} - (a.\nabla)\vec{E} = 0\\
&\vec{\theta}\times\vec{B} - \vec{\varphi}\times\vec{E} + (\vec{x}.\vec{\varphi})\frac{\partial\vec{B}}{\partial t} + \left(t\vec{\varphi} + \vec{x}\times\vec{\theta}\right)\frac{\partial}{\partial\vec{x}}\vec{B} - (a.\nabla)\vec{B} = 0
\end{aligned}\right\} \quad (A.3)$$

Then, let us write the equations corresponding to fields invariant under J_z, K_z, J_ξ and A_x.

Defining the quantities $\begin{cases} F_\pm = E_x \pm B_y \\ G_\pm = E_y \pm B_x \end{cases}$

we obtain

α) Invariance under J_z :

Taking then $\vec{\varphi} = 0$ and $\vec{\theta} = (0,0,1)$ in (A.3), we deduce :

$$\left.\begin{aligned}
&G_+ - DF_- = 0\\
&G_- - DF_+ = 0\\
&F_- + DG_+ = 0\\
&F_+ + DG_- = 0\\
&DE_z = DB_z = 0
\end{aligned}\right\} \qquad \text{with } D = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}$$

β) Invariance under K_z :

we take $\vec{\varphi} = (0,0,1)$ and $\vec{\theta} = 0$, it follows

$$\left.\begin{aligned}
&F_+ - DF_+ = 0\\
&F_- + DF_- = 0\\
&G_+ + DG_+ = 0\\
&G_- - DG_- = 0\\
&DE_z = DB_z = 0
\end{aligned}\right\} \qquad \text{with } D = \eta\frac{\partial}{\partial\eta} - \zeta\frac{\partial}{\partial\zeta}$$

if $\eta = t+z$, $\zeta = t-z$.

γ) Invariance under $J_\xi = \cos\frac{\xi}{\ell}J_z - \sin\frac{\xi}{\ell}K_z$

$$\left.\begin{aligned}
&G_- + \varphi F_+ - DF_+ = 0\\
&G_+ + \varphi F_- + DF_- = 0\\
&F_- + \varphi G_+ + DG_+ = 0\\
&F_+ + \varphi G_- + DG_- = 0\\
&DE_z = DB_z = 0
\end{aligned}\right\}$$

with $D = \varphi\left(\eta\frac{\partial}{\partial\eta} - \zeta\frac{\partial}{\partial\zeta}\right) - \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ and $\varphi = \frac{\xi}{\ell}$

δ) Invariance under $A_x = K_x - J_y$

$$\left.\begin{aligned}
&\ell E_z + DF_+ = 0\\
&\ell B_z + DG_- = 0\\
&F_- - DE_z = 0\\
&G_+ - DB_z = 0\\
&DG_+ = DF_- = 0
\end{aligned}\right\} \qquad \text{with } D = \ell x\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial x}$$

- APPENDIX B -

ELECTROMAGNETIC PLANE WAVES AND SYMMETRY

Let us restrict ourselves to progressive electromagnetic plane waves, i.e. electromagnetic fields occuring in the vacuum in the absence of charges and depending only of the $t-z=\xi$ variable, if we take the z-axis as the direction of propagation of the fields.

Then such a field is invariant (at least) under $G\{T^0(3)\}$ and we deduce immediately from the Maxwell equations :

$$\left.\begin{aligned} E_z &= c^t \\ B_z &= c^t \\ F_- &= E_x - B_y = c^t \\ G_+ &= E_y + B_x = c^t \end{aligned}\right\}$$

Studying now separately each connected Poincaré subgroup G of dimension less or equal to six and such that $T_c = T^0(3)$ we can say that the groups $G\{T^0(3), T_z\}$ and $G\{T^0(3), T_z, K_z\}$ are sub-c.e.P.s. of $G\{T(4), T_z, K_z\}$ while $G\{T^0(3), A_x + P_t + P_z, A_y\}$ is a sub-c.e.P.s. of $G\{T(4), A_x, A_y\}$. $G\{T^0(3), A_x\}$ is a sub-c.e.P.s. of $G\{T^0(3), A_x, A_y\}$ and $G\{T^0(3), K_z, A_x\}$ is a sub-c.e.P.s. of $G\{T^0(3), K_z, A_x, A_y\}$. The fields

solutions of these c.e.P.s. have been given in section IV as well as the solutions of the c.e.P.s. $G\{T^0(3), T_t, A_x, A_y\}$ and $G\{T^0(3), A_x, A_y, T_z + \varepsilon(P_t + P_z)\}$, $\varepsilon = \pm 1$. Finally the respective general solutions of $G\{T^0(3), T_z + \varepsilon(P_t + P_z)\}$, $\varepsilon = \pm 1$, $G\{T^0(3), K_z\}$ and $G\{T^0(3), T_t\}$ differ from the respective solutions of $G\{T^0(3), A_x, A_y, T_z + \varepsilon(P_t + P_z)\}$ $\varepsilon = \pm 1$, $G\{T^0(3), K_z, A_x, A_y\}$ and $G\{T^0(3), T_t, A_x, A_y\}$ given in section IV only by the fact that the components E_z and B_z of the fields invariant under these four-dimensional c.e.P.s. respectively are not necessarily zero, but can have non zero constant values.

Before ending this Appendix, it is worth noticing that in the general case where we do not impose the field F to propagate in the vacuum in absence of charges, invariance under $G\{T^0(3)\}$ implies, using the Maxwell equations :

$$j_x = j_y = 0 \quad ; \quad -\frac{\partial E_z}{\partial t} = j_z = \rho \qquad \text{setting} \quad \rho = j_0$$

Discording the unphysical case $j_z = \rho = e^t$, we remark that ρ and j_z verify the wave equation

$$\frac{\partial^2 \rho}{\partial t^2} - \frac{\partial^2 \rho}{\partial z^2} = 0$$

and so correspond to charged particles with zero mass.

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P A R T I E III

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LES ALGEBRES DE SCHRODINGER, GALILEE ET CONFORME EN MECANIQUE QUANTIQUE

CHAPITRE I :

ON THE EXPLICIT-TIME DEPENDENT INVARIANCE PROPERTIES OF QUANTUM MECHANICAL SYSTEMS

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ABSTRACT : We use the notion of split extension algebra to embody a given invariance algebra of a quantum mechanical system, a realization of which is known in terms of some variables, in a richer invariance algebra expressed in terms of the same variables. By applying this procedure to a free system of particles we show how to obtain the invariance under the Schrödinger algebra and we build a bigger invariance algebra which describes a system of non-interacting particles, for example the asymptotic states in a non relativistic scattering problem.

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INTRODUCTION

As long as we do not take into account an explicit time dependence, the notion of symmetry algebra for a conservative system is defined by the property that all its own generators commute with the Hamiltonian of the problem and furnish constants of motion. In the case of explicitly time dependent transformations it has been shown [1] that the corresponding generators leave invariant the Hamiltonian of the Schrödinger equation iff :

$$[H, S(t)] = i\,\partial S(t)/\partial t \qquad (1)$$

Lipkin noted that if ψ is an eigenfunction of H, $S\psi$ cannot be an eigenfunction of H with the same eigenvalue (in the case where the equation (1) is not trivially satified). Hence S(t) satifying Eq.(1) generates families of states having different energies and provides informations about the energy spectrum of H : a time-dependent symmetry algebra works as a spectrum generating algebra, which increases the own interest of such a concept.

Morecver it is easy to see that all the explicit time derivatives of S(t) satifying Eq.(1) furnish invariance properties too. In the simplest non trivial case where S(t) is a linear function of the time, $\partial S / \partial t$ is time-independent and commutes with H. This is the case for the generators of the pure Galilean transformations $S_j(t) \equiv K_j = M R_j - P_j t$ ($j = 1,2,3$) which verify:

$$[H, K_j] = -iP_j \qquad [H, P_j] = 0$$

Translational invariance generated by P_j is then automatically involved by the requirement of pure Galilean invariance generated by K_j. "It is impossible to consider the former without the later" said Lipkin and the constraints brought on the interaction by the invariance under Galilean "boosts" imply those required by the invariance under translations. Then this remark emphasizes the interest to work as far as possible with large invariance algebras.

It appears ,from this short discussion,that the concept of explicit time dependent symmetry makes coexist two mathematical notions : links between some generators of the invariance algebra by integration with respect to time,and correspondingly particular positions of certain subalgebras in the considered invariance algebra .

At this step,it has seemed natural to us to consider the following problem : being given an elementary invariance algebra associated to an Hamiltonian describing a physical system ,is it possible to embody this algebra into a bigger invariance one? It is a aim of this paper to answer this question. In fact the hypothesis of the problem can be enlarged and we propose here an algebraic method to embody an elementary invariance algebra for an unspecified Hamiltonian into a larger time dependent invariance algebra a realization of which can be given in terms of the same observables used for the realization of the starting algebra.

This paper will be divided into three parts :

-In section 2 we discuss the correspondence between invariance properties in the Schrödinger's picture and non-explicit time dependent observables in the Heisenberg's picture.

-In section 3 we show how the mathematical frame furnished by the notion of derivation algebra can be used to increase a given invariance algebra called a "germ" in our terminology.

- Section 4 deals with two applications of the above method.By choosing as germs two subalgebras of the extended Galilean algebra we generate as invariance algebra the Schrödinger algebra and a bigger one which appears as the smallest invariance algebra able to describe a system of non-interacting particles.

2- TIME-DEPENDENT INVARIANCE ALGEBRA IN THE SCHRODINGER AND HEISENBERG PICTURES.

It is easy to deduce from Eq.(1) that the commutator $\left[S_1, S_2\right] = S_1 S_2 - S_2 S_1$ for any invariance generators S_1 and S_2 is an invariance generator too, hence the invariance properties of a given Hamiltonian form a Lie algebra. Now Eq.(1) does not imply it is a finite dimensional Lie algebra, but we restrict ourselves to finite one for mathematical convenience.

Let us therefore consider a finite N-dimensional invariance algebra. In Schrödinger picture each generator is satisfying the relations:

$$\left[H\ ,\ S_j(t)\right] = i\,\partial S_j / \partial t = i \sum_{k=1}^{N} \phi_j^{\,k}\, S_k(t) \qquad \forall\ j = 1, \ldots N \qquad (2)$$

where the coefficients $\phi_j^{\,k}$ are time-independent.

Such generators possess the following important property : they do not depend explicitly on time in the Heisenberg picture. Indeed in going from the Schrödinger representation to the Heisenberg one, any operator Ω changes according to the rule:

$$\Omega_H = e^{iHt}\ \Omega\ \ e^{-iHt} \qquad (3)$$

and for any generator verifying (1) it is easy to see that :

$$\partial S_H / \partial t = e^{iHt}\left(i\left[H\ ,\ S\right] + \partial S / \partial t \right) e^{-iHt} = 0$$

Consequently S_H does not depend explicitly on time and, from Eq.(3), takes its value for $t = 0$. We shall write :

$$S_H = S(0) = S_o \qquad (4)$$

Hence the generators of an invariance algebra do not depend explicitly upon time in the Heisenberg picture but do not commute necessarily with H. More precisely the generators S_o generate an algebra $\mathcal{S}_o$ isomorphic to the starting invariance algebra $\mathcal{S}(t)$ since each S_o deduces from a corresponding S by means of the automorphism defined in Eq.(3).

Conversely to any algebra of observables which do not depend explicitly on time corresponds an invariance algebra isomorphic to it in the Schrödinger

picture. Indeed the explicit time derivative of S(t) defined by :

$$S(t) = e^{-iHt}\, S_0\, e^{iHt} \tag{5}$$

gives exactly the invariance condition (1).

Obviously each coefficient in the development of S(t) in powers of t

$$S(t) = S_0 - it\,[H, S_0] + \ldots + (-it)^m/m!\,.\underbrace{[H[H[H \ldots [H}_{m \text{ times}}, S_0]\ldots]] + \ldots \tag{6}$$

is a generator of the algebra $\mathcal{A}_0$. Then it should be noticed that in the case where the development is boundless, the explicit time dependence of the corresponding S(t) is an exponential one. But if there exists m such that $\underbrace{[H[H \ldots [H}_{m+1}, S_0]\ldots] = 0$, S(t) is a polynomial of highest degree m in t. In this last case the coefficient of t^m is identical in both pictures since it commutes with H.

It is worth noticing that this last results should be obtained directly from the right hand equation contained in (2) which furnishes a system of N linear differential equations of the first order with constant coefficients when j runs from 1 to N. The resolution of this system determines the explicit time dependence of the $S_j(t)$ and leads to the above properties.

As it was already noticed in the Introduction, chains of elements can be obtained in the algebra $\mathcal{A}(t)$ by successive derivations with respect to the time : $S(t)$, $\partial S(t)/\partial t$, $\partial^2 S(t)/\partial t^2$, ... but also by repeated action of ad(H) if one remembers Eq.(1). Obviously the isomorphism between $\mathcal{A}(t)$ and $\mathcal{A}_0$ insures that the same phenomenon exists in the algebra $\mathcal{A}_0$, but in the Heisenberg picture the chains of elements can only be obtained by repeated action of ad(H). Hence in practice the relationships between generators appear when the whole algebra is decomposed under the action of H.

We conclude this section by remarking that from a mathematical point of view the algebras $\mathcal{A}_0$ and $\mathcal{A}(t)$ are the same algebra depending on t as a real parameter. An interesting consequence is that the Casimir operators of this algebra are time independent, such an example is given in reference [2].

3- EXTENSION OF A GIVEN INVARIANCE ALGEBRA.

We start with a well defined invariance algebra : the germ $\mathcal{G}_0$ a realization of which is given in the Heisenberg picture in terms of some non-commuting observables ω_j ($j = 1,\ldots n$) used to describe a quantum mechanical system. We wish to embody this germ into a bigger algebra able to yield new invariance properties in terms of the same variables.Now we insist on the fact that our aim is not to extend the germ by another invariance algebra expressed in terms of other observables which was the case when one tried to extend the Poincare algebra by internal symmetry such as SU(3) for example.

In our approach the Hamiltonian is not given explicitly and we have in mind to exhibit the constraints on H which come from the embedding of the germ into a larger invariance algebra.To solve this problem it is not sufficient to give a realization of a larger algebra chosen in the enveloping algebra $\mathcal{E}(\Theta)$ of the algebra Θ generated by the observables ω_j .Indeed in this way one doesn't know the action of the larger algebra on H.In the same mind it is not possible to extend the germ by time integration of some generators, that is to introduce new operators such that $Z_j = \int K_j \, dt = Z_{oj} + MR_j t - \frac{1}{2}P_j t$ in the example of the introduction.In this case the commutation relations with H are well defined : $[H, Z_j] = iK_j$ but we have not construct a bigger invariance algebra since commutators like $[K_j, Z_k]$ and $[P_j, Z_k]$ are not determined It is then necessary to build an algebraic scheme which provides an algebra $\mathcal{F}$ containing the germ $\mathcal{G}_0$ and making appear the commutation relations of H with the new introduced generators.

By hypothesis $\mathcal{G}_0$ is a subalgebra of $\mathcal{E}(\Theta)$ and the largest subalgebra of $\mathcal{F}$ which can be set in isomorphism with a subalgebra of $\mathcal{E}(\Theta)$ will be a candidate to the wanted extended algebra $\mathcal{L}_0$ .The explicit time realisation i.e. the invariance algebra $\mathcal{L}(t)$ is then obtained by the automorphism defined in Eq.(5).

In order to construct the above mentioned Lie algebra $\mathcal{F}$ we shall use a mathematical tool closely connected to the structure of a given Lie algebra which is the Lie algebra of its automorphisms : the derivation algebra. This notion is very natural and strongly suggested by the fact that in section 2 we have already considered H as a derivation acting on the invariance algebra. On the other hand it is very simple to extend a Lie algebra by its derivation algebra*. Indeed let be $\mathcal{A}$ a finite dimensional Lie algebra and $\mathcal{D}(\mathcal{A})$ the derivation algebra, it is then possible to construct the split extension of $\mathcal{D}(\mathcal{A})$ by $\mathcal{A}$, also called the holomorph of $\mathcal{A}$, which is the semi-direct sum $\mathcal{A} \oplus \mathcal{D}(\mathcal{A})$ [3]. The elements of this new algebra are the ordered pairs (a,d) where a and d belong respectively to $\mathcal{A}$ and $\mathcal{D}(\mathcal{A})$, and the Lie product is defined by :

$$\left[(a,d)\ ,\ (a',d')\right] = \left([a,a'] + d(a') - d'(a)\ ,\ [d,d']\right) \qquad (7)$$

This construction gives a meaning to the brackets $[d,a']$, indeed :

$$[d,a'] = \left[(0,d)\ ,\ (a',0)\right] = (d(a')\ ,\ 0) = d(a') \in \mathcal{A} \qquad (8)$$

which is just the action of the derivations on the elements of the initial algebra.

Let us apply now this construction to a given germ $\mathcal{G}_o$. In general the Hamiltonian belongs to the germ and the suggested method is able to make appear new generators d the action of which on H is not trivial, i.e. $d(H) = S_o$ with $S_o \in \mathcal{G}_o$ and $S_o \neq 0$. Therefore they correspond to time integrals of generators of the germ in the Schrödinger picture, but it should be noticed that we have not succeeded to exhibit an extension giving rise to integration of Galilean boosts.

Now it happens that the Hamiltonian doesn't appear alone or accompanied in the derived algebra of the given invariance algebra. It is then possible to exclude H out of the germ and, owing to its action on the invariance algebra, we are sure to identify H in the derivation algebra. We will show in the next section that this proposal is not purely academic but allows to show some interesting links between some invariance algebras.

4 - APPLICATIONS TO A FREE SYSTEM OF PARTICLES

In general the Galilei group is considered as the full kinematical invariance group of the Hamiltonian of a free system, but it has been recently shown by Niederer [4] that a larger group of space time transformations : the Schrödinger group, leaves invariant the free Schrödinger equation. In fact it has been shown [2] that this embedding of the Galilei group strongly limits the form of the interaction between the constituent particles of the system. We propose to apply the technic above described to generate the Schrödinger algebra and moreover to exhibit a larger invariance algebra which appears as a limit case in the description of a system of particles, since no any interaction can take place between them.

We recall that a system of particles can be set in correspondence with an UNIRREP of a central extension $\widetilde{G}$ of the Galilei group, the generator realizations of which, are given, in terms of canonical co-ordinates $q_j(\mu)$ and $p_j(\mu)$ corresponding to each mass point $m(\mu)$, $\mu = 1,2,\dots,N$, by

$$P_j = \sum_{\mu=1}^{N} p_j(\mu) \quad , \quad M = \sum_{\mu=1}^{N} m(\mu) \quad , \quad J_j = \sum_{\mu} \left(\underline{q}(\mu) \wedge \underline{p}(\mu) \right)_j \qquad (9)$$

$$K_j = \sum_{\mu} m(\mu)\, q_j(\mu) - P_j t = M X_j - P_j t \quad , \quad H = \sum_{\mu=1}^{N} \frac{p^2(\mu)}{2m_\mu} + V = T + V$$

The non-zero commutation relations are given by :

$$\left[J_j , J_k \right] = i \varepsilon_{jkl} J_l \quad , \quad \left[J_j , P_k \right] = i \varepsilon_{jkl} P_l \quad , \quad \left[J_j , K_k \right] = i \varepsilon_{jkl} K_l$$

$$\left[K_j , P_k \right] = i \delta_{jk} M \quad , \quad \left[H , K_j \right] = -i P_j \qquad (10)$$

A) - Let us choose as a first example the derived extended Galilei group generated by $\{K_j, P_j, J_j, M\}$, $j = 1,2,3$, i.e. we take as germ the corresponding Lie algebra $\widetilde{g}'_0$ in the Heisenberg picture.

As it has been shown in ref.[5] the derivation algebra $\mathcal{D}(\tilde{\mathcal{G}}'_o)$ contains the subalgebra of derivations isomorphic to the derived non-extended Galilean algebra and four supplementary generators which form between them a subalgebra isomorphic to $\mathcal{Gl}(2,\mathbb{R})$. The problem consists now in finding in the enveloping associative algebra $\mathcal{E}(\underline{q}(\mu),\underline{p}(\mu))$ the largest subalgebra isomorphic to a subalgebra of $\mathcal{F} = \tilde{\mathcal{G}}'_o \oplus \mathcal{D}(\tilde{\mathcal{G}}'_o)$ and including $\tilde{\mathcal{G}}'_o$. It is easy to see that we cannot realize simultaneously the whole inner derivation algebra, and if we consider only a subalgebra, the realization coïncides with that of the corresponding generators of $\tilde{\mathcal{G}}'_o$ and there is a redundance. Now the subalgebra $\mathcal{Gl}(2,\mathbb{R})$ can be decomposed into the direct sum $\mathcal{SU}(1,1) \oplus \mathcal{R}_{D_2}$, but the one-parameter subalgebra $\mathcal{R}_{D_2}$ acts as a dilatation on M and obviously cannot be set in correspondence with any element of $\mathcal{E}(\underline{q},\underline{p})$ and then must be rejected.

Among the $\mathcal{SU}(1,1)$ generators, one of them can be recognized, owing to its action on $\tilde{\mathcal{G}}'$, as the Hamiltonian of the system. The others, denoted C_o and D_o, are given by :

$$C_o = \frac{1}{2}\sum_{\mu=1}^{N} m(\mu) q^2(\mu)$$
$$D_o = -\frac{1}{2}\sum_{j=1}^{3}\sum_{\mu=1}^{N}\left[q_j(\mu), p_j(\mu)\right]_+ \qquad (11)$$

Their commutation relations are :

$$\left[H, C_o\right] = i D_o \qquad \left[H, D_o\right] = 2i H$$
$$\left[C_o, D_o\right] = -2i C_o \qquad (12)$$

And by using the transformation (5) we obtain the explicit time-dependent generators of the extended invariance algebra in the Schrödinger picture :

$$D = 2Ht + D_o \quad , \quad C = Ht^2 + D_o t + C_o \qquad (13)$$

In fact this invariance algebra obtained from the derivations of $\tilde{\mathcal{G}}'$ is the Schrödinger algebra $\mathcal{S}_4 = \tilde{\mathcal{G}}' \oplus \mathcal{SU}(1,1)$, the commutation relations of which are given by (10,12) and following (8) by the non-trivial actions of the derivations on the form :

$$\left[D, K_j\right] = i K_j \quad , \quad \left[D, P_j\right] = -i P_j \quad , \quad \left[C, P_j\right] = i K_j \qquad (14)$$

B) - As second example we will consider a subalgebra in the preceding germ namely the Heisenberg algebra $\mathcal{H}_0$ generated by $\{K_j, P_j, M\}$. The loss of the rotational invariance i.e. of the semi-simple subalgebra generated by J, leads to a richer derivation algebra [5]. Indeed the inner derivations form a 6-dimensional Abelian algebra and we obtain a basis of $\mathcal{D}(\mathcal{H}_0)$ by adding 22 generators which generate a subalgebra isomorphic to $\left(\mathcal{Sp}(6,\mathbb{R}) \oplus \mathcal{R}_{D_2}\right)_0$. For the same reasons as previously, we must neglect in $\mathcal{H}_0 \square \mathcal{D}(\mathcal{H}_0)$ the generators corresponding to the inner derivations and to the dilatation of the mass D_2 . Finally it is possible to give a realization of the semi-direct sum $\mathcal{A}_{b_0} = \mathcal{H}_0 \square \left(\mathcal{Sp}(6,\mathbb{R})\right)_0$. Now it appears a new feature : the Hamiltonian is fully determined and cannot contain any interaction : $H = \sum_\mu \frac{p^2(\mu)}{2m_\mu}$. The realization of a convenient basis of the $\left(\mathcal{Sp}(6,\mathbb{R})\right)_0$ algebra is given by :

$$
\begin{aligned}
R_{jk} &= \frac{1}{2} \sum_{\mu=1}^{n} \left[p_j(\mu) \, , \, q_k(\mu) \right]_+ \qquad j,k = 1,2,3 \\
S_{jk} &= \sum_{\mu=1}^{n} m(\mu)\, q_j(\mu)\, q_k(\mu) = S_{kj} \qquad (15) \\
T_{jk} &= \sum_{\mu=1}^{n} \frac{p_j(\mu)\, p_k(\mu)}{m(\mu)} = T_{kj}
\end{aligned}
$$

It is easy to see that $\mathcal{A}_{b_0}$ contains the extended Schrödinger algebra and we have the following relations in terms of the generators of the above basis :

$$
\begin{aligned}
J_j &= -\varepsilon_{jke} R_{ke} \quad , \quad H = \frac{1}{2} \sum_{j=1}^{3} T_{jj} \\
C_0 &= \frac{1}{2} \sum_{j=1}^{3} S_{jj} \quad , \quad D_0 = -\sum_{j=1}^{3} R_{jj}
\end{aligned}
\qquad (16)
$$

The non-zero commutation relations of $\mathcal{A}_{b_0}$ are given by :

$$\left[R_{jk}, R_{\ell m}\right] = i\left(\delta_{k\ell} R_{jm} - \delta_{jm} R_{\ell k}\right)$$

$$\left[R_{jk}, T_{\ell m}\right] = i\left(\delta_{k\ell} T_{jm} + \delta_{km} T_{j\ell}\right)$$

$$\left[R_{jk}, S_{\ell m}\right] = -i\left(\delta_{j\ell} S_{km} + \delta_{jm} S_{k\ell}\right) \qquad (17)$$

$$\left[S_{jk}, T_{\ell m}\right] = i\left(\delta_{j\ell} R_{mk} + \delta_{jm} R_{\ell k} + \delta_{k\ell} R_{mj} + \delta_{km} R_{\ell j}\right)$$

$$\left[R_{k\ell}, K_{oj}\right] = -i\,\delta_{jk} K_{o\ell} \quad , \quad \left[R_{k\ell}, P_j\right] = i\,\delta_{j\ell} P_k$$

$$\left[S_{k\ell}, P_j\right] = i\left(\delta_{jk} K_{o\ell} + \delta_{j\ell} K_{ok}\right) \quad , \quad \left[T_{k\ell}, K_{oj}\right] = -i\left(\delta_{jk} P_\ell + \delta_{j\ell} P_k\right)$$

Owing to the inclusion above mentioned $\mathcal{H}_{o} \subset \mathcal{A}s_{o}$ it remains only to give the explicit time dependence of the last 15 generators. Following the method described in Sect.2 we decompose $\mathcal{A}s_{o}$ under the action of H and from (17) firstly we obtain :

$$\left[H\,,\, S_{jk}\right] = -i\left(R_{jk} + R_{kj}\right), \left[H\,,\, R_{jk} + R_{kj}\right] = -2i\, T_{jk} \quad (j \neq k) \qquad (18)$$

Then the transformation (5) furnishes the three following generators :

$$Q_{jk} = T_{jk}\, t^2 - \left(R_{jk} + R_{kj}\right) t + S_{jk} \quad , \quad j \neq k \qquad (19)$$

and also the six explicit time derivatives $\dot{Q}_{jk}$ and $\ddot{Q}_{jk}$.

The two last chains correspond to the commutation relations which determine H and which can be written under the convenient but redundent following form :

$$\left[H, S_{jj} - \tfrac{1}{3}\sum_{k=1}^{3} S_{kk}\right] = -2i\left(R_{jj} - \tfrac{1}{3}\sum_{k} R_{kk}\right)$$

$$\left[H, R_{jj} - \tfrac{1}{3}\sum_{k} R_{kk}\right] = -2i\left(T_{jj} - \tfrac{1}{3}\sum_{k} T_{kk}\right) = -2i\left(T_{jj} - \tfrac{1}{3} H\right) \qquad (20)$$

of which we deduce two independent generators

$$Q_{jj} = \left(T_{jj} - \tfrac{1}{3}\sum_{k} T_{kk}\right) t^2 - 2\left(R_{jj} - \tfrac{1}{3}\sum_{k} R_{kk}\right) t - \left(S_{jj} - \tfrac{1}{3}\sum S_{kk}\right) \qquad (21)$$

such that $\sum_{j=1}^{3} Q_{jj} = 0$, $j=1,2,3$, and their explicit time derivatives.

In summary we will notice that our approach allowed us to build the $\mathcal{A}_5$ algebra which is the invariance algebra of a free system of non-interacting particles. As we have seen $\mathcal{A}_5$ contains the Schrödinger algebra which is the largest algebra of space-time transformations which leaves invariant the free Schrödinger equation ; in fact, as it has been shown in [2], the Schrödinger invariance still allows to introduce some types of interaction between particles, while the $\mathcal{A}_5$ algebra, which comes from the derivations of the Heisenberg algebra, does not offer such a possibility and can uniquely describe a system of particles without interaction.

CONCLUSION.

The technics we proposed allows to construct the following chain of algebras : $\mathcal{H} \subset \tilde{\mathcal{G}} \subset \tilde{\mathcal{P}}_2 \subset \mathcal{A}_5$.
In this chain the number of invariance generators increase while the internal interaction melts. As a direct consequence of this property it may be asked what kinds of external fields are compatible with the above invariance algebras ? We will treat this subject in a forthcoming paper.

Finally we emphasize the property of the algebra $\mathcal{A}_5 = \mathcal{H} \square \mathcal{Sp}(6,R)$ which really appears as the candidate to describe the asymptotic states in non-relativistic scattering problems.

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LETTERE AL NUOVO CIMENTO VOL. 7, N. 16 18 Agosto 1973

CHAPITRE II :

Invariance Algebras and Interactions in Nonrelativistic Quantum Mechanics.

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(ricevuto il 4 Giugno 1973)

In a previous paper (1) we have studied the possibilities of extending a « germ », that is, a given invariance algebra of a quantum-mechanical system, and we have proposed a method using the notion of derivation algebra. In particular we have treated, as an example, a system of particles possessing Galilean invariance. In this case the Hamiltonian appears as a derivation which acts on the remainder of the algebra, *i.e.* the extended derived Galilean algebra $\tilde{\mathfrak{g}}'$. Hence by taking $\tilde{\mathfrak{g}}'$ as germ, our approach has permitted us to realize the whole Schrödinger algebra $\widetilde{\mathrm{Sch}} = \tilde{\mathfrak{g}}' \square su_{1,1}$, in which the Hamiltonian belongs to the $su_{1,1}$ subalgebra. This approach has led us to ask the following question. Is it possible to extract from the $SU_{1,1}$ algebra other types of Hamiltonians which, by construction, act as a derivation on $\tilde{\mathfrak{g}}'$ and describe systems which always possess $\tilde{\mathfrak{g}}'$ as invariance subalgebra? By answering to the question we will show that it appears an interesting algebraic characterization of the notion of interaction (external fields or internal interactions among constituents of a system).

A system of N nonrelativistic particles without spin is characterized in part by the centre-of-mass observables: the position $\boldsymbol{X} = \boldsymbol{K}/M$ and the canonically conjugate variable $\boldsymbol{P}$. In the Heisenberg picture these observables can be expressed in terms of canonical co-ordinates $\boldsymbol{q}(\mu)$ and $\boldsymbol{p}(\mu)$ corresponding to each mass-point $m(\mu)$ $(\mu = 1, \ldots, N)$ following

$$(1) \qquad P_{0j} = \sum_{\mu=1}^{N} p_j(\mu) , \qquad K_{0j} = \sum_{\mu=1}^{N} m(\mu) q_j(\mu) , \qquad M = \sum_{\mu} m(\mu) \qquad \text{with } j = 1, 2, 3 ,$$

and we have the following nonzero commutation relations (Heisenberg algebra $\mathscr{H}$):

$$(2) \qquad [K_{0j}, P_{0k}] = i\delta_{jk} M .$$

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(1) G. BURDET, M. PERRIN and P. SORBA: *On the explicit-time-dependent invariance properties of a quantum-mechanical system*, preprint Marseille 72/P. 499 (1972).

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Further, if we suppose that the system possesses rotational invariance, we are led to introduce the SO_3 algebra generated by

$$J_j = \sum_{\mu=1}^{N} (\boldsymbol{q}(\mu) \wedge \boldsymbol{p}(\mu))_j \tag{3}$$

and the semi-direct sum $\mathscr{H} \square so_3$ furnishes the wanted germ isomorphic to the derived extended Galilean algebra $\tilde{\mathfrak{g}}'$. The commutation relations are given by

$$[J_j, J_k] = i\varepsilon_{jkl} J_l\,, \qquad [J_j, P_{0k}] = i\varepsilon_{jkl} P_{0l}\,, \qquad [J_j, K_{0k}] = i\varepsilon_{jkl} K_{0l}\,. \tag{4}$$

By using the notion of split extension it has been shown in ref. (1) that, among the derivations of $\tilde{\mathfrak{g}}'$, only a $SU_{1,1}$ algebra provides new invariance generators. The algebra $\tilde{\mathfrak{g}}' \square su_{1,1}$ obtained by this way is isomorphic to the Schrödinger algebra known as the Lie algebra of the largest group of co-ordinate transformations leaving invariant the free Schrödinger equation (2). Owing to its action on $\tilde{\mathfrak{g}}'$, the algebra $su_{1,1}$ is realized in terms of quadratic quantities in $\boldsymbol{p}(\mu)$ and $\boldsymbol{q}(\mu)$ and a convenient basis is given by

$$T = \sum_{\mu} \boldsymbol{p}(\mu)^2/2m(\mu)\,, \qquad C_0 = \tfrac{1}{2} \sum_{\mu} m(\mu)\boldsymbol{q}(\mu)^2, \qquad D_0 = -\tfrac{1}{2} \sum_{j=1}^{3} \sum_{\mu} [q_j(\mu), p_j(\mu)]_+ \tag{5}$$

with the commutation relations

$$[T, C_0] = iD_0\,, \qquad [T, D_0] = 2iT\,, \qquad [D_0, C_0] = 2iC_0\,. \tag{6}$$

The nontrivial actions of these derivations on the germ lead to the following commutation relations:

$$[D_0, K_j] = iK_j\,, \qquad [D_0, P_j] = -iP_j\,, \qquad [C_0, P_j] = iK_j\,, \qquad [T, K_j] = -iP_j\,. \tag{7}$$

Now it is well known that $su_{1,1}$ ($\simeq sl_{2,R} \simeq so_{2,1} \simeq sp_{2,R}$) possesses only one two-dimensional subalgebra up to a conjugation and three one-dimensional nonconjugate subalgebras.

Let $\beta C_0 + \gamma D_0 + \delta T$ (β, γ, δ real) be the general element of $su_{1,1}$, then the one-dimensional subalgebras can be classified according as $\alpha^2 = \gamma^2 - \beta\delta$ is positive, negative or null.

The situation is summarized in Fig. 1, where we put in each box a typical example of generators of nonconjugate subalgebras.

Hence it is possible to select three different classes of Hamiltonians among the derivations of $\tilde{\mathfrak{g}}'$ corresponding to the three one-dimensional subalgebras. The general form of these Hamiltonians will be

$$H = T + \beta C_0 + \gamma D_0 + V\,, \tag{8}$$

where V must act as a trivial derivation on $\tilde{\mathfrak{g}}'$: $[V, \tilde{\mathfrak{g}}'] = 0$ and so does not modify the action of the general element of $su_{1,1}$ on the germ. Thus it is always possible to put in

(1) U. NIEDERER: *Helv. Phys. Acta*, **45**, 802 (1972); P. ROMAN, J. J. AGHASSI, R. SANTILLI and P. HUDDELSTON: *Nuovo Cimento*, **12** A, 185 (1972).

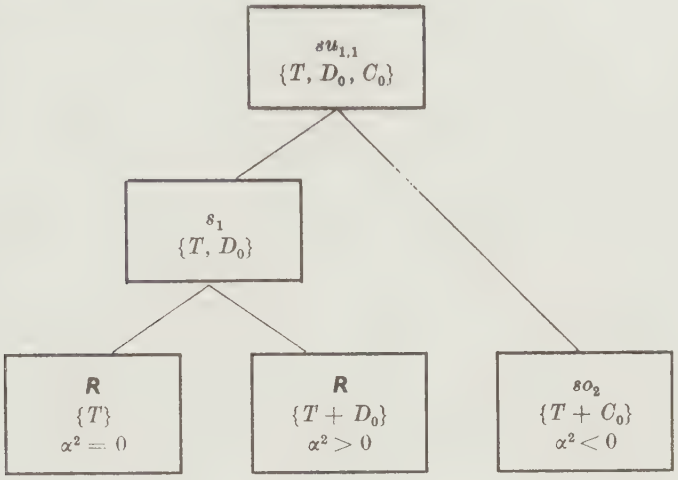

Fig. 1.

H the familiar Galilean internal interaction V in addition to the external potentials C_0 and D_0. It is well known that such a V only depends on the scalars under rotations built from the relative vectors

$$\boldsymbol{q}(\mu, \nu) = \boldsymbol{q}(\mu) - \boldsymbol{q}(\nu) \quad \text{and} \quad \boldsymbol{p}(\mu, \nu) = \frac{m(\nu)\boldsymbol{p}(\mu) - m(\mu)\boldsymbol{p}(\nu)}{m(\mu) \cdot m(\nu)} .$$

The action of H on the germ is deduced from (7):

$$[H, \boldsymbol{J}] = 0 , \quad [H, \boldsymbol{K}_0] = i(\gamma \boldsymbol{K}_0 - \boldsymbol{P}_0) , \quad [H, \boldsymbol{P}_0] = i(\beta \boldsymbol{K}_0 - \gamma \boldsymbol{P}_0) . \tag{9}$$

Up to now we have worked in the Heisenberg picture, but the notion of invariane algebra is defined in the Schrödinger picture by the following property: all its own generators $S(t)$ satisfy

$$[H, S(t)] = i \frac{\partial S(t)}{\partial t} . \tag{10}$$

Then it is interesting to go in the Schrödinger picture by the automorphism

$$S_0 \to \exp[-iHt] S_0 \exp[iHt] = S(t) \qquad \text{for any } S_0 \in \tilde{\mathfrak{g}}' .$$

So we obtain

$$\begin{cases} \boldsymbol{K} = \left(\dfrac{\gamma}{\alpha} \sinh \alpha t + \cosh \alpha t\right) \boldsymbol{K}_0 - \dfrac{1}{\alpha} \sinh \alpha t \cdot \boldsymbol{P}_0 , \\ \boldsymbol{P} = \left(-\dfrac{\gamma}{\alpha} \sinh \alpha t + \cosh \alpha t\right) \boldsymbol{P}_0 + \dfrac{\beta}{\alpha} \sinh \alpha t \cdot \boldsymbol{K}_0 , \end{cases} \quad \text{if } \alpha^2 = \gamma^2 - \beta \neq 0 , \tag{11a}$$

$$\begin{cases} \boldsymbol{K} = (1 + \gamma t) \boldsymbol{K}_0 - t \boldsymbol{P}_0 , \\ \boldsymbol{P} = (1 - \gamma t) \boldsymbol{P}_0 + \gamma^2 t \boldsymbol{K}_0 , \end{cases} \quad \text{if } \alpha^2 = 0 . \tag{11b}$$

By hypothesis the angular momentum commutes with H, hence is identical in the two pictures.

We return now to the general form (8) of the Hamiltonian, which in all cases can be written

$$H = T_\gamma - \alpha^2 C_0 + V \quad \text{with} \quad T_\gamma = T + \gamma^2 C_0 + \gamma D_0 = \mu \frac{(\boldsymbol{p}(\mu) - \gamma m(\mu)\boldsymbol{q}(\mu))^2}{2m(\mu)} . \tag{12}$$

But it is easy to see that

$$\exp[i\gamma C_0]\, T_\gamma \exp[-i\gamma C_0] = T . \tag{13}$$

Therefore it is always possible to reduce to T the kinematical-energy term T_γ. The conjugation defined in (13) acts as a gauge transformation and each of the three classes of Hamiltonians is defined up to a specific external vector potential.

Among the various algebras $\tilde{\mathfrak{g}}' \square H$ it is interesting to remark that for $\alpha^2 = 0$ we obtain an algebra isomorphic to the extended Galilean one $\tilde{\mathfrak{g}}$, which corresponds to $\gamma = 0$.

In the class $\alpha^2 < 0$ the algebra can be obtained as a central extension of a contraction ($c \to \infty$) of the De Sitter algebra $so_{3,2}$ [3]. We notice that H generates the compact so_2 subalgebra, hence possesses a discrete spectrum and for $\gamma = 0$, $\beta = \omega^2$, $\alpha = i\omega$ we find the usual harmonic-oscillator Hamiltonian.

The adjunction to the germ of a two- or three-dimensional algebra brings only supplementary constraints on the trivial derivation V which can be included in the Hamiltonian. We briefly mention that the adjunction of a two-dimensional algebra makes appear the condition $[D_0, V] = -2iV$ already studied in ref. [4], then V can only be a linear function of a small number of arguments the dependence of which on the canonical variables is well defined.

Finally the consideration of the whole algebra $su_{1,1}$ adds the condition $[V, C_0] = 0$, which eliminates any $\boldsymbol{p}(\mu, \nu)$ dependence in V, and ensures gauge invariance under the transformation (13). Now V can still be a linear function of the interesting two-, three- and four-body terms:

$$\frac{1}{\boldsymbol{q}(\mu, \nu)^2}, \qquad \frac{q(\mu, \nu)\cdot q(\mu, \lambda)}{\boldsymbol{q}(\mu, \nu)^2 \boldsymbol{q}(\mu, \lambda)^2}, \qquad \frac{q(\mu, \nu)\cdot q(\varrho, \lambda)}{\boldsymbol{q}(\mu, \nu)^2 \boldsymbol{q}(\varrho, \lambda)^2} .$$

The following commutation relations are deduced from (6):

$$[H, C_0] = i(2\gamma C_0 + D_0) , \qquad [H, D_0] = 2i(H - 2\beta C_0 - \gamma D_0) . \tag{14}$$

They allow to write in the Schrödinger picture

$$\begin{cases} C = \left(\cosh 2\alpha t + \dfrac{\gamma}{\alpha}\sinh 2\alpha t\right) C_0 + \dfrac{1}{2\alpha}\sinh 2\alpha t D_0 + \dfrac{1}{2\alpha^2}(\cosh 2\alpha t - 1) H , \\ D = \left(\cosh 2\alpha t - \dfrac{\gamma}{\alpha}\sinh 2\alpha t\right) D_0 - 2\dfrac{\beta}{\gamma}\sinh 2\alpha t C_0 + \\ \qquad + \left(\dfrac{\gamma}{\alpha^2} + \dfrac{1}{\alpha}\sinh 2\alpha t - \dfrac{\gamma}{\alpha^2}\cosh 2\alpha t\right) H , \end{cases} \tag{15a}$$

(3) A. SUDBERY: *Nucl. Phys.*, **44** B, 520 (1972).
(4) G. BURDET and M. PERRIN: *Lett. Nuovo Cimento*, **4**, 651 (1972).

for $\alpha^2 \neq 0$, and

$$(15b) \qquad \begin{cases} C = (1 + 2\gamma t)C_0 + tD_0 + t^2 H\,, \\ D = (1 - 2\gamma t)D_0 - 4\gamma^2 tC_0 + 2t(1 - \gamma t)H\,, \end{cases} \qquad \text{if } \alpha^2 = 0\,.$$

In ref. ([4]) the case of a free system of interacting particles was studied and the corresponding explicit-time realization of the Schrödinger algebra deduces from (11*b*) and (15*b*) with $\gamma = 0$.

Another particular case has been studied in ref. ([5]), it is the case of one particle in an external harmonic-oscillator field, which corresponds to an Hamiltonian belonging to the class characterized by $\alpha^2 < 0$, and the explicit-time realization comes from (11*a*) and (15*a*) with $\beta = \omega^2$, $\gamma = 0$, $\alpha = i\omega$.

In conclusion, a method using the notions of derivation algebra and split extension has permitted us ([1]) to extend the extended derived Galilean algebra $\tilde{\mathfrak{g}}'$ by a $su_{1,1}$ algebra and to construct the semi-direct sum $\tilde{\mathfrak{g}}' \square su_{1,1}$ isomorphic to the extended Schrödinger algebra $\widetilde{\mathrm{Sch}}$. Pursuing the study of this example in the present paper without limiting ourselves to a free system of particles, we have shown that

to each one-dimensional nonconjugate subalgebra of the simple Lie algebra $su_{1,1}$ can be associated a class of Hamiltonians,

these Hamiltonians contain some external fields which belong to the realized derivations acting effectively on the germ,

it is also possible to include in the Hamiltonians mutual interactions between the constituents of the system, provided that these interactions act as a trivial derivation on $\tilde{\mathfrak{g}}'$; moreover they are submitted to some constraints imposed by the inclusion in the added algebra.

In fact these various points which emerge from a simple example are quite general and it appears very promising to study in the same way nonrelativistic systems which possess no longer rotational invariance. We shall then have at our disposal a richer derivation algebra than previously and shall be led to consider various external fields such as anisotropic harmonic oscillator fields, angular-momentum and quadrupole-momentum coupling. This will be the aim of a subsequent paper.

([5]) U. NIEDERER: *The maximal kinematical-invariance group of the harmonic oscillator*, preprint Institut für Theoretische Physik der Universität (Zurich, 1972).

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CHAPITRE III :

About the Non-relativistic Structure of the Conformal Algebra

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Abstract. A new basis of the conformal algebra is proposed, which makes appear two conjugated Schrödinger algebras. This basis allows to exhibit a chain which does not contain the Poincaré algebra, between the $\mathscr{SC}(4,2)$ algebra and the (two-dimensional) extended Galilean one. This non-relativistic structure of the conformal algebra is well adapted to discuss some extreme models of hadrons based on collinear massless particles.

The existence of an isomorphism between a subgroup of the Poincaré group and the two-dimensional Galilean group is a well-known fact which clearly appeared in the "infinite-momentum" or "light-cone" formalism of high energy physics and led a "potential theoretic intuition" in relativistic quantum field theory and quantum electrodynamics.

More recently this Galilean structure in the "transverse plane" was used to derive the phenomenological Parton model of Björken and Feynman [1].

In this paper we show that the non-relativistic structure of the transverse plane again emerges in the conformal algebra which contains a subalgebra isomorphic to the two-dimensional Schrödinger algebra. We deduce that to any relativistic massless particle can be associated a non-relativistic one having a variable mass but without the potential energy present in the massive relativistic case.

Moreover we show that the two-dimensional Schrödinger algebra is the stability algebra of the "color" of the lightlike states introduced by Domokos [2] and we discuss the extreme collinear free massless quarks model of hadrons considered by Del Giudice *et al.* [3].

It is well-known that the d'Alembert equation is invariant under the conformal group which contains as a subgroup the Poincaré transformations, invariance group of the Klein-Gordon equation. The conformal Lie algebra is isomorphic to the $\mathscr{SC}(4,2)$ algebra generated by the 15 skew-symmetric $M_{\mu\nu}(\mu,\nu=0,0',1,2,3,4)$ satisfying the following commutation relations:

$$[M_{\mu\nu},M_{\varrho\sigma}]=i(g_{\mu\sigma}M_{\nu\varrho}+g_{\nu\varrho}M_{\mu\sigma}-g_{\nu\sigma}M_{\mu\varrho}-g_{\mu\varrho}M_{\nu\sigma}) \qquad (1)$$

with

$$g_{00} = g_{0'0'} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1 .$$

The generally used basis of the conformal algebra makes appear:

— A Lorentz algebra generated for example by the $M_{\mu\nu}$ such that $\mu, \nu = 0, 1, 2, 3$.

— The corresponding translational and special conformal Abelian algebras

$$p_\mu = M_{0'\mu} + M_{4\mu} \quad \text{and} \quad c_\mu = M_{0'\mu} - M_{4\mu} \quad \text{where} \quad \mu = 0, 1, 2, 3$$

which respectively form with $\mathscr{SO}(3, 1)$ two conjugated Poincaré algebras.

— The dilatation $M_{0'4}$ which commutes with the Lorentz part.

In fact another decomposition of the $\mathscr{SO}(4, 2)$ algebra can be given which exhibits its non-relativistic structure.

First one finds a subalgebra isomorphic to the two-dimensional extended Galilean algebra $\tilde{\mathscr{G}}_2$, the situation of which in the Poincaré algebra has been widely used. The generators are:

$$\begin{aligned} &H = \tfrac{1}{2}(p_0 - p_3) = \tfrac{1}{2}(M_{0'0} + M_{40} - M_{0'3} - M_{43}), \qquad J_3 = M_{12} \\ &P_j = p_j = M_{0'j} + M_{4j}, \qquad K_j = -(M_{0j} + M_{3j}), \qquad j = 1, 2 \\ &M = p_0 + p_3 = M_{0'0} + M_{40} + M_{0'3} + M_{43} \end{aligned} \tag{2}$$

with the following non-zero commutation relations

$$\begin{aligned} &[J_3, P_j] = i\varepsilon_{3jk} P_k, \qquad [J_3, K_j] = i\varepsilon_{3jk} K_k, \qquad j, k = 1, 2 \\ &[H, K_j] = -iP_j, \qquad [K_j, P_k] = i\delta_{jk} M . \end{aligned} \tag{3}$$

Recall in particular that $\{K_j, P_j, M\}$ form a two-dimensional Heisenberg algebra $\mathscr{H}_2$, and that this choice of the generators of $\tilde{\mathscr{G}}_2$ corresponds to the usual change of basis used in the infinite momentum formalism.

Secondly, $\tilde{\mathscr{G}}_2$ is embedded in the larger two-dimensional extended Schrödinger algebra $\tilde{\mathscr{S}}_2$ which corresponds to the largest group of the Newtonian space-time transformations leaving invariant the two-dimensional Schrödinger equation [4]. This algebra is 9-dimensional and a basis can be given by adding to (2) the two generators

$$\begin{aligned} &C = \tfrac{1}{2}(c_0 + c_3) = \tfrac{1}{2}(M_{0'0} - M_{40} + M_{0'3} - M_{43}) \\ &D = M_{0'4} - M_{03} . \end{aligned} \tag{4}$$

We have to add to (3) the following commutation relations

$$\begin{aligned} &[D, P_j] = -iP_j, \qquad [D, K_j] = iK_j, \qquad [D, H] = -2iH , \\ &[C, P_j] = iK_j, \qquad [C, H] = -iD, \quad [D, C] = 2iC . \end{aligned} \tag{5}$$

It must be noticed that H, C, D generate an algebra isomorphic to $\mathscr{SU}(1, 1)$.

Finally we can complete a basis of the $\mathcal{SO}(4,2)$ algebra with another Heisenberg algebra $\mathcal{H}_2^*$ generated by

$$\begin{aligned} &P_j^* = M_{0j} - M_{3j}, \quad K_j^* = c_j - M_{0'j} - M_{4j}, \\ &M^* = c_0 - c_3 = M_{0'0} - M_{40} - M_{0'3} + M_{43} \end{aligned} \tag{6}$$

and a 15th generator $\Lambda = -(M_{0'4} + M_{03})$ which plays the role of a dilatation or scale transformation

$$\begin{aligned} &[\Lambda, P_j] = iP_j, \quad [\Lambda, P_j^*] = -iP_j^*, \\ &[\Lambda, K_j] = iK_j, \quad [\Lambda, K_j^*] = -iK_j^*, \quad j = 1,2, \\ &[\Lambda, M] = 2iM, \quad [\Lambda, M^*] = -2iM^*. \end{aligned} \tag{7}$$

Between the two Heisenberg algebras one finds the commutation relations

$$\begin{aligned} &[P_j, P_k^*] = 2i\delta_{jk}H, && [K_j, K_k^*] = 2i\delta_{jk}C, \\ &[P_j, K_k^*] = -2i\varepsilon_{jk3}J_3 - i\delta_{jk}(D-\Lambda), && [P_j^*, K_k] = 2i\varepsilon_{jk3}J_3 + i\delta_{jk}(D+\Lambda), \\ &[P_j, M^*] = 2iP_j^*, && [K_j, M^*] = 2iK_j^*, \\ &[P_j^*, M] = -2iP_j, && [K_j^*, M] = -2iK_j, \end{aligned} \tag{8}$$

$$[M, M^*] = -4i\Lambda.$$

Let us remark that M, M^*, Λ also generate a 3-dimensional non-compact simple algebra. It is not necessary to write the commutation relations between the algebra $\mathcal{SO}(2) \oplus \mathcal{SU}(1,1)$ ($\mathcal{SO}(2)$ generated by J_3) and $\mathcal{H}_2^*$ because $\mathcal{SO}(2) \oplus \mathcal{SU}(1,1)$ acts on $\mathcal{H}_2^*$ exactly as on $\mathcal{H}_2$ and in fact one can form the two Schrödinger algebras

$$\begin{aligned} \tilde{\mathcal{S}}_2 &= \mathcal{H}_2 \,\square\, (\mathcal{SO}(2) \oplus \mathcal{SU}(1,1)), \\ \tilde{\mathcal{S}}_2^* &= \mathcal{H}_2^* \,\square\, (\mathcal{SO}(2) \oplus \mathcal{SU}(1,1)). \end{aligned}$$

These two algebras are conjugated, which can be proved by considering the action of $\mathcal{SO}(4,2)$ on the pseudo-euclidian space $E(4,2)$ and in fact turns to introduce the algebra

$$\mathcal{ISO}(4,2) = \mathcal{T}(4+2) \,\square\, \mathcal{SO}(4,2).$$

$\mathcal{T}(4+2)$ is the 6-dimensional translational algebra and $\mathcal{SO}(4,2)$ acts on its generators V_μ as

$$[M_{\mu\nu}, V_\varrho] = i(g_{\nu\varrho}V_\mu - g_{\mu\varrho}V_\nu). \tag{9}$$

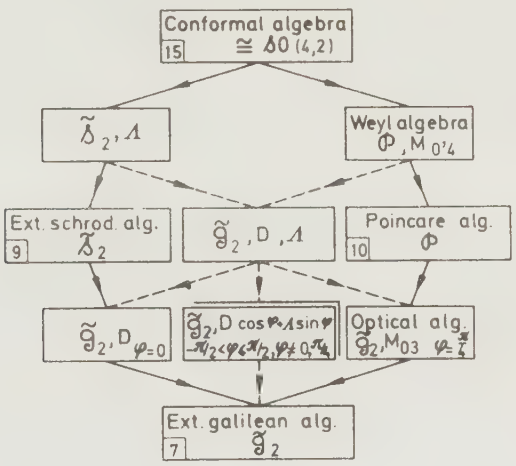
Then it can be shown that the 10-dimensional algebra $(\tilde{\mathcal{S}}_2, \Lambda)$ is the stabilizer of the 2-plane $(V_{0'} + V_4, V_0 + V_3)$. Indeed one can verify that

$(\tilde{\mathscr{P}}_2, A)$ is maximal in $\mathscr{SO}(4,2)$ and by using (9) that it stabilizes the above 2-plane. Identically $(\tilde{\mathscr{P}}_2^*, A)$ is the stabilizer of the 2-plane $(V_0 - V_4, V_0 - V_3)$.

One achieves the proof by checking that the two above 2-planes are conjugated [for example under the inner automorphism: $V \to \exp(i\pi M_{34}) \cdot V \exp(-i\pi M_{34})$] which ensures that $\tilde{\mathscr{P}}_2$ and $\tilde{\mathscr{P}}_2^*$ are conjugated in the same way.

Hence the decomposition we have introduced permits to show the existence of an interesting chain of subalgebras between $\tilde{\mathscr{G}}_2$ and $\mathscr{SO}(4,2)$ which does not contain the Poincaré algebra and we can draw the following complete chains (we mean by complete a chain such that it does not exist any algebra between two adjacent algebras).

To be more precise we have represented by dotted lines the little chains which exist between the two physical chains, and it is worth noticing that there exists no other chain between $\tilde{\mathscr{P}}_2$ and $\mathscr{SO}(4,2)$.



Physically the consequence of the inclusion of $\tilde{\mathscr{P}}_2$ in $\mathscr{SO}(4,2)$ leads to associate to any relativistic massless particle a non-relativistic one with a variable mass in a two-dimensional space (the transverse plane). Indeed the d'Alembert equation $p^2|\rangle = 0$, describing the states $|\rangle$ of a massless relativistic particle in the momentum space, formally becomes

$$\left(H - \frac{P_\perp^2}{2M}\right)|\rangle = 0 \quad \text{with} \quad P_\perp = (P_1, P_2), \quad \text{for states such that} \quad M|\rangle \neq 0,$$

i.e., furnishes the Schrödinger equation describing the states of a free non-relativistic particle which is invariant under the Schrödinger algebra.

Furthermore if we start with two non-relativistic massless particles a and b, then $p = p(a) + p(b)$ and we are led to the formal Schrödinger equation

$$\left(H - \frac{P_\perp^2(a)}{2M(a)} - \frac{P_\perp^2(b)}{2M(b)} + V\right)|\rangle = 0$$

where V is a function of the relative transverse momentum $P_\perp(a, b)$

$$V = \frac{M(a) + M(b)}{2M(a)\,M(b)}\, P_\perp^2(a, b)$$

with

$$P_\perp(a, b) = \frac{M(a)P_\perp(b) - M(b)P_\perp(a)}{M(a) + M(b)}.$$

This does not contradict the invariance under $\tilde{\mathscr{S}}_2$ since it has been shown in Ref. [5] that this invariance allows the existence of some classes of mutual interactions between the constituents of the free system.

In fact the existence of such a non-relativistic structure in the conformal algebra is subjacent in the wave packet realization of lightlike states given by Domokos [2].

In our formalism the definition of a lightlike state becomes

$$P_j|\rangle = 0, \quad j = 1, 2,$$
$$M|\rangle = k|\rangle$$

where k is the color of the state. From the above commutation relations (3, 5, 7, 8) it is clear that the algebra of transformations which leave the color invariant, i.e. the algebra which commutes with M is exactly $\tilde{\mathscr{S}}_2$. The Domokos's lightlike states completely agree with our interpretation of the conformal algebra since the color is associated to the "mass" of the non-relativistic particle, but we must remark that the generator of "time" translations (the "Hamiltonian" H) does not correspond to the compact subalgebra of $\mathscr{SU}(1, 1)$: $H + C$ alone generates $\mathscr{SO}(2)$, hence is able to have a discrete oscillatorlike spectrum [5].

The situation is rather different in what concerns the extreme collinear free massless quarks model introduced in Ref. [3]. The collinear states describing massless particles travelling in the z-direction are defined up to a conjugation by

$$P_j|\rangle = 0, \quad j = 1, 2,$$
$$H|\rangle = 0.$$

But the used invariance notion of state space makes the model go out of our interpretation and leads to consider the algebra which in our formalism corresponds to J_3, D, M, M^*, Λ. As it has been above noticed

M, M^*, A form a non-compact algebra which corresponds to $\mathcal{O}(2,1)$ in Ref. [3], and it may be interesting to remark that their conformal quantum number is the eigenvalue of the compact generator $M - M^*$ which turns to account the "masses" of the two Heisenberg algebras present in the decomposition of $\mathcal{SO}(4,2)$ we have given.

We hope that this brief analysis brings some insight about the point raised in the footnote (9) of the Ref. [3] in the following sense: the Domokos's approach uses the underlying non-relativistic analogy contained in the conformal algebra, while the invariance notion retained by Del Giudice *et al.* discards this analogy.

In conclusion we lay stress on the scheme we have drawn which clearly makes appear the double, – relativistic and non-relativistic –, content of the conformal algebra and the conformal nature of the Schrödinger algebra.

Finally let us add that the same kind of scheme can be drawn between the algebra $\mathcal{SO}(5,2)$ and the 3-dimensional Galilean algebra embedded on one side in the Schrödinger algebra $\tilde{\mathcal{G}}_3$, on the other side in the Poincaré algebra $\mathcal{P}(4,1)$ corresponding to transformations on the $\mathcal{M}(4,1)$ Minkowski space.

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CHAPITRE IV

CONNECTED SUBGROUPS OF THE GALILEI AND EXTENDED GALILEI GROUPS

P. SORBA

Talk presented at the 3rd International Colloquium on Group Theoretical Methods in Physics, June 1974, Centre de Physique Théorique, CNRS, Marseille.

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0 - Introduction.

This talk is devoted to the study of the "Connected subgroups of the Galilei and extended Galilei groups" [1] . In this work, all the connected subgroups of the Galilei group G - and of its central extension $\tilde{G}$ - are determined and classified up to a conjugation of G - respectively $\tilde{G}$ - and also up to an isomorphism.

As an introduction, one could say that this classification can be inserted into a set of works, or can appear as a consequence of some previous works.

Indeed, two years ago the classification of the Poincaré subgroups was achieved by H. Bacry, Ph. Combe and myself [2] . Then it might appear interesting to apply the same treatment to the Galilei group . In fact, for the classification of the Galilei subgroups, some techniques constructed in view of the classification of the Poincaré subgroups have been used, but it has been also possible to improve some of them, or to construct new tools.

Let me mention that the classification of Poincaré subgroups allowed us to determine and to give a classification of electromagnetic fields carrying as a symmetry a subgroup of the Poincaré group [3] .

I must add a study we have done with G. Burdet and M. Perrin, which has led to a series of papers [4] , and which can be summarized under the title : "Quantum Mechanics around the Schrödinger algebra and the inhomogeneous symplectic one" . M. Perrin has to present, during this Colloquium, certain of these results and also new results [5] she has just obtained with G. Burdet. In this study, the importance of the Galilean algebra structure and the role of certain Galilean subalgebras have incited me to continue the classification of Galilei subgroups.

Another common point links the different points mentioned above : indeed all the works they represent have been - or are - presented in one of the three Colloquia on Group Theoretical Methods in Physics between 1972 and 1974.

Here are selected some results which, I hope, furnish a general landscape of the set of the Galilei connected subgroups, the complete classification being given in Ref. [1] .

I - Generalities.

a) First of all, let us recall :

Theorem :- If H is a Lie subgroup of G, its Lie algebra $\mathcal{H}$ is a subalgebra of the Lie algebra $\mathcal{G}$ of G.

- Each subalgebra of $\mathcal{G}$ is the Lie algebra of exactly one connected Lie subgroup of G.

It follows from this theorem that the classification of all the connected subgroups of a given connected Lie group G is given by the classification of all the Lie subalgebras of $\mathcal{G}$.

Classification of the connected subgroups of G $\Longleftrightarrow$ Classification of the Lie subalgebras of $\mathcal{G}$.

b) It is clear that we are interested in subgroups up to a conjugation.

In fact, the subgroups (subalgebras) of G ($\mathcal{G}$) and $\tilde{G}$ ($\tilde{\mathcal{G}}$) are classified + up to a conjugation of G ($\tilde{G}$).
+ up to an isomorphism.

c) Let us remark that if the set of the subalgebras of a Lie algebra $\mathcal{G}$ is a lattice, the set of the conjugacy classes is not (in general) a lattice, but only a partially ordered set $\mathcal{N}(\mathcal{G})$ (order = inclusion).

II - Structure of the Galilean algebra $\mathcal{G}$ (and of $\tilde{\mathcal{G}}$).

a) Commutation Relations (C.R.) of $\mathcal{G}$ ($\tilde{\mathcal{G}}$) :

We recall the non zero C.R. of $\mathcal{G}$ and $\tilde{\mathcal{G}}$, denoting by J_j $(j=1,2,3)$ the rotation generators, K_j the Galilean boosts, P_j the space-translation and H the time-translation generators [6] :

$$[J_j, J_k] = i\,\varepsilon_{jkl}\,J_l \quad ; \quad [J_j, K_k] = i\,\varepsilon_{jkl}\,K_l \qquad (j,k,l=1,2,3).$$

$$[J_j, P_k] = i\,\varepsilon_{jkl}\,P_l$$

$$[H, K_j] = -i\,P_j$$

for $\tilde{\mathcal{G}}$: $$[K_j, P_l] = i\,\delta_{jl}\,m\mathbf{I} = i\,\delta_{jl}\,M.$$

b) Remarkable decompositions of $\mathcal{G}$ ($\tilde{\mathcal{G}}$):

It is interesting to remark the following decompositions of $\mathcal{G}$ (and $\tilde{\mathcal{G}}$)* :

i) $\mathcal{G} = \{\vec{P}, \vec{K}, H\} \supset \{\vec{J}\}$: Levi's decomposition [7].

ii) $\mathcal{G} = \{\vec{P}, H\} \supset \{\vec{J}, \vec{K}\}$. : this decomposition may recall us the Inönü-Wigner contraction from the Poincaré algebra : $\mathcal{P} = \mathcal{T} \supset \mathcal{L}$ into the Galilean one, contraction in which the Lorentz algebra $\mathcal{L}$ is transformed into the euclidean algebra $E_3 = \{\vec{J}, \vec{K}\}$.

iii) $\mathcal{G} = \{\vec{P}, \vec{K}, \vec{J}\} \supset \{H\} = \mathcal{G}' \supset \{H\}$

$\mathcal{G}'$ being the derived algebra, also called isochronous Galilean algebra.

iv) $\mathcal{G} = \{\vec{K}, \vec{P}\} \supset \{\vec{J}, H\}$.

Similar decompositions can be given for the extended Galilean algebra $\tilde{\mathcal{G}}$. Denoting $\mathcal{H}(3)$ the Heisenberg algebra $\{\vec{P}, \vec{K}, M\}$, the last decomposition becomes : $\tilde{\mathcal{G}} = \mathcal{H}(3) \supset \left[\{\vec{J}\} \oplus \{H\}\right]$. We can note the inclusion

$$\tilde{\mathcal{G}} \subset \widetilde{\mathcal{S}chröd} = \mathcal{H}(3) \supset \left[\{\vec{J}\} \oplus \{H, C, D\}\right] \subset \mathcal{H}(3) \supset \mathcal{S}p(6,R)$$

$\widetilde{\mathcal{S}chrod}$. denoting the extended Schrödinger algebra, $\mathcal{S}p(6,R)$ the symplectic algebra acting on the 6-dimensional phase space.

All the C.R. of $\widetilde{\mathcal{S}chröd}$. will be known by adding to the C.R. of $\tilde{\mathcal{G}}$ the following non-zero ones :

$$[C, P_j] = -i K_j \qquad [C, H] = -i D$$

$$[D, K_j] = i K_j \qquad [D, H] = -2i H$$

$$[D, P_j] = -i P_j \qquad [D, C] = 2i C$$

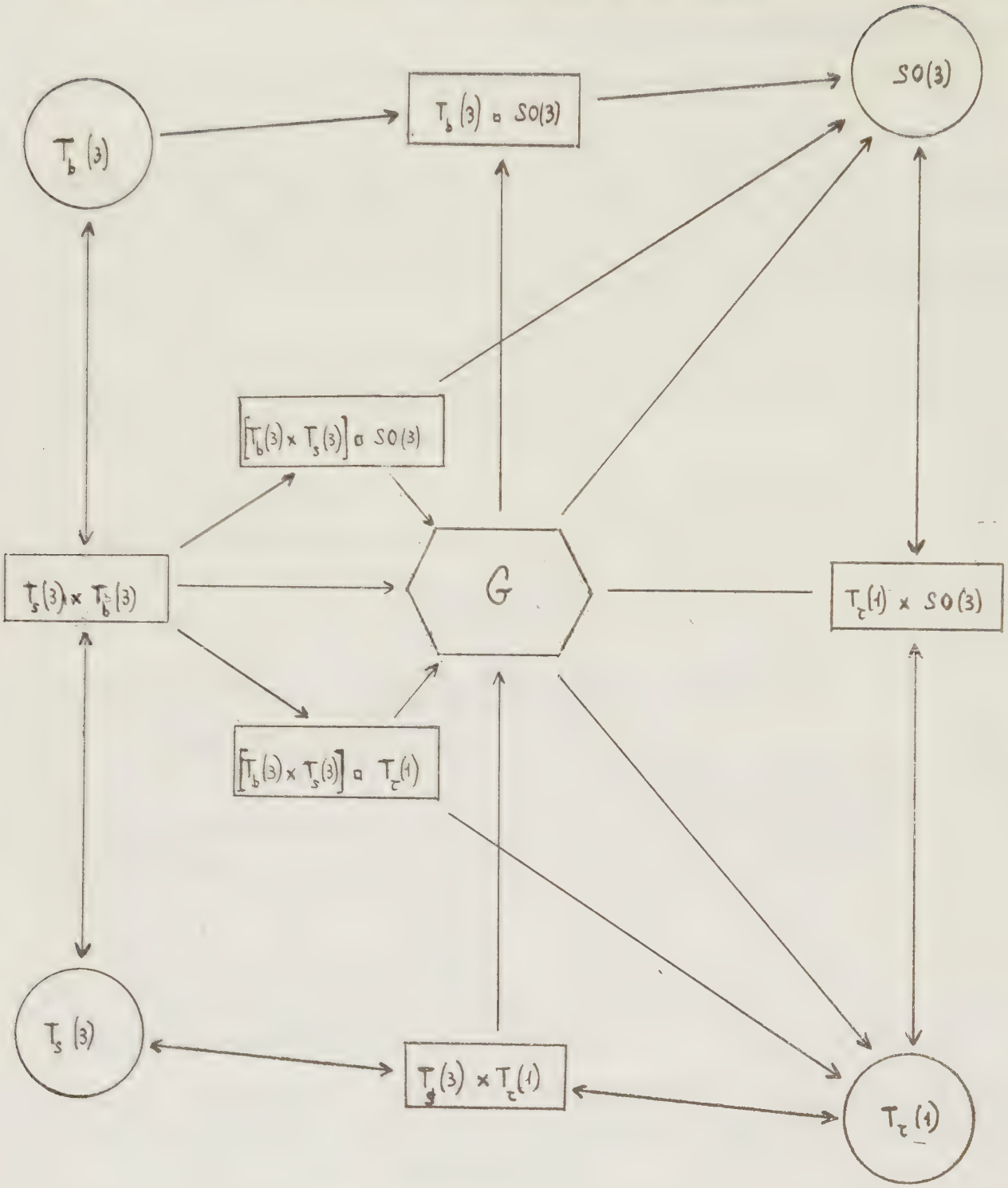
A part of this discussion is summarized in table 1**.

---

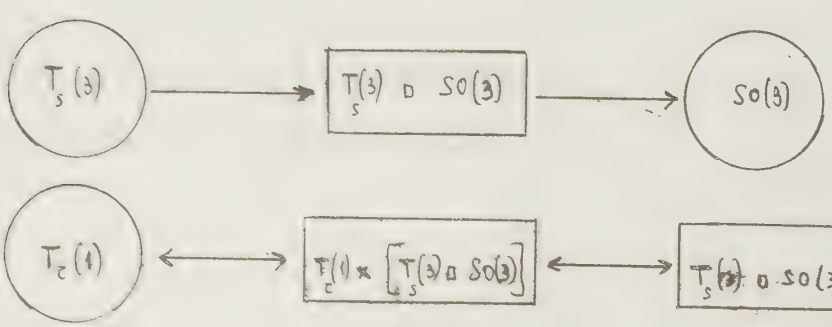
* We denote by $\{X, Y, \ldots Z\}$ the Lie algebra a basis of which is given by $X, Y \ldots Z$.

** The subscripts τ, s, b mean respectively time, space and boost.

Table I . Exact Sequences and Subgroups of G [8]



Two subgroups do not appear in these decompositions :



III - General Remarks on Subalgebras of $\mathcal{G}$ ($\tilde{\mathcal{G}}$).

a) Proposition : Let A, B and C be Lie groups and let $C = B \supset A$. Then two elements of A are conjugate under a transformations of C iff they are conjugate under a transformation of A.

As a consequence, the p.o.s. $\mathcal{H}(A)$ is a subset of $\mathcal{H}(C)$.

Application : We deduce that :

$\mathcal{H}(\{\vec{J}, \vec{K}\})$
$\mathcal{H}(\{\vec{J}, H\})$ are subsets of $\mathcal{H}(\mathcal{G})$.
$\mathcal{H}(\{\vec{J}\})$
$\mathcal{H}(\{H\})$

b) Ideals of $\mathcal{G}$:

Property : An interesting property of an ideal is that it is its own conjugacy class. We note that : $\{\vec{J}, \vec{K}, \vec{P}\}$
$\{H, \vec{K}, \vec{P}\}$
$\{\vec{K}, \vec{P}\}$
$\{\vec{P}\}$ are ideals of $\mathcal{G}$.

c) Euclidean algebras $E(3)$ in $\mathcal{G}$:

There are 2 non conjugate euclidean algebras $E_b(3)$ & $E_s(3)$.
$\{\vec{J}, \vec{K}\}$ $\{\vec{J}, \vec{P}\}$.

It can be shown that $\mathcal{H}(E_b(3))$ and $\mathcal{H}(E_s(3))$ are subsets of $\mathcal{H}(\mathcal{G})$.

d) Two dimensional subalgebras in $\mathcal{G}$ ($\tilde{\mathcal{G}}$) :

Property : All two dimensional subalgebras of $\mathcal{G}$ ($\tilde{\mathcal{G}}$) are abelian.

IV - Maximal subalgebras of a real Lie algebra.

a) Theorem : Let $\mathcal{A}$ be a real Lie algebra, a Levi's decomposition of which being $\mathcal{A} = \mathcal{R} \supset \mathcal{L}$; where $\mathcal{R}$ is the radical of $\mathcal{A}$ and $\mathcal{L}$ a Levi's factor. Then each maximal subalgebra $\mathcal{B}$ of $\mathcal{A}$ can be written, up to a conjugation either as : $\mathcal{B} = \mathcal{R} \supset \mathcal{L}_B$ where $\mathcal{L}_B$ is some maximal subalgebra of $\mathcal{L}$.
or as : $\mathcal{B} = \mathcal{R}_B \supset \mathcal{L}$ where $\mathcal{R}_B$ is some subalgebra of $\mathcal{R}$, maximal invariant subalgebra under $\mathcal{L}$.

It follows that every maximal subalgebra $\mathcal{B}$ of $\mathcal{A}$ is symmorphic.

The proof is given in Ref. [1] . In fact, this theorem generalizes several properties proved in Ref. [2].

Let us retall here the definition of a "symmorphic subalgebra" : so let $\mathcal{B}$ be a subalgebra of $\mathcal{A}$: $\mathcal{B} = \mathcal{L}_B + \mathcal{M}_B + \mathcal{R}_B$ (as vector-space)

$\mathcal{M}_B$: complementary subspace of $\mathcal{L}_B + \mathcal{R}_B$ in $\mathcal{B}$.

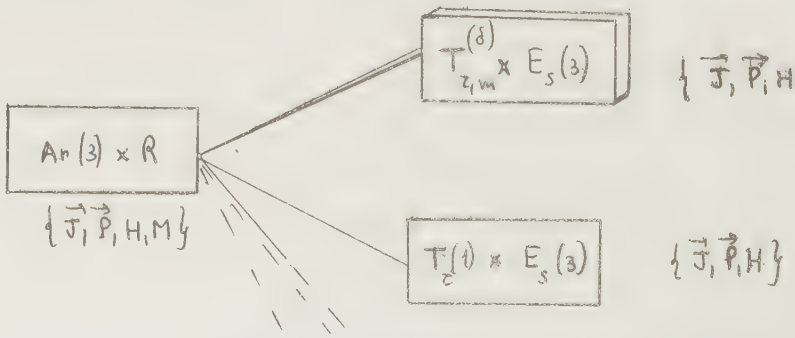
$m \in \mathcal{M}_B \Rightarrow m' = l' + r'$ $\quad l' \in \mathcal{L}$; $\notin \mathcal{L}_B$
$m \neq 0$ $\quad r' \in \mathcal{R}$; $\notin \mathcal{R}_B$
$l' \neq r'$ non zero.

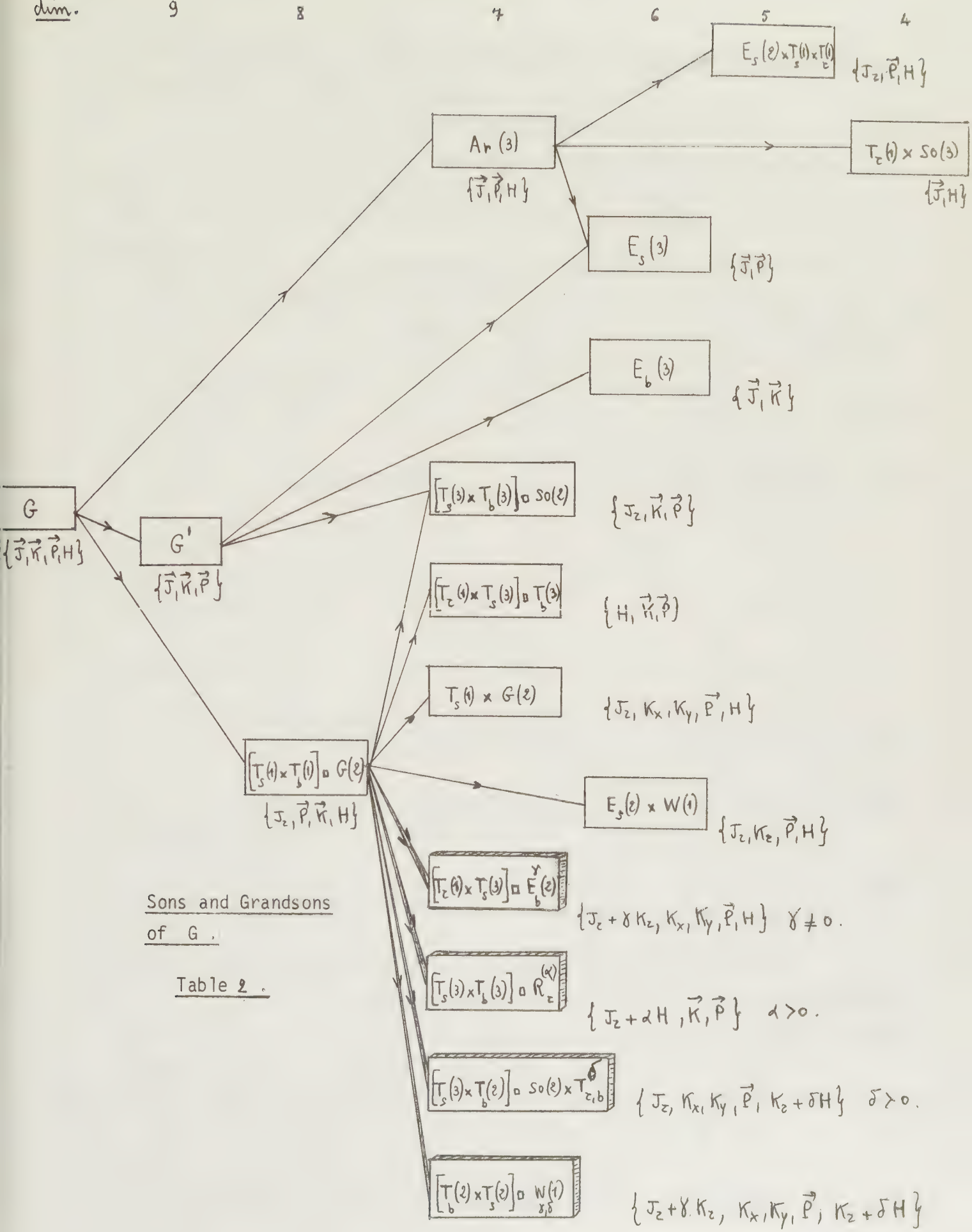
By definition, $\mathcal{B}$ is said symmorphic if $\mathcal{M}_B = 0$ or if $\mathcal{B}$ is conjugate to $\mathcal{B}'$ such that $\mathcal{M}_{B'} = 0$.

b) Sons and grandsons of $\mathcal{G}$ (and $\tilde{\mathcal{G}}$) :

Using the previous theorem, it is easy to deduce the maximal subalgebras (sons) and the maximal subalgebras of the maximal subalgebras (grandsons) of $\mathcal{G}$: see table 3 .

Sons and grandsons of $\tilde{\mathcal{G}}$ can be deduced by the same way, or by using techniques proposed in § V . Practically, the table of sons and grandsons of $\tilde{\mathcal{G}}$ can be obtained by adding the generator M of the extension to each subalgebra of table 2 , and also by drawing two new boxes corresponding to subalgebras of $Ar(3) \oplus R$:





Sons and Grandsons of G .

Table 2 .

V - Relations between subalgebras of $\mathcal{G}$ and of $\tilde{\mathcal{G}}$.

- the p.o.s $\mathcal{N}(\tilde{\mathcal{G}})$ can be deduced from $\mathcal{N}(\mathcal{G})$ without much difficulty owing to the following remarks.

- Let us consider the exact sequence :

$$0 \xrightarrow{\alpha} \mathcal{M} \xrightarrow{\mu} \tilde{\mathcal{G}} \xrightarrow{\theta} \mathcal{G} \xrightarrow{\beta} 0$$

$\mathcal{M}$ being the one dimensional subalgebra generated by the extension generator M

Then to each subalgebra $\tilde{\mathcal{H}} \subset \tilde{\mathcal{G}}$ can be associated the subalgebra : $\theta(\tilde{\mathcal{H}}) = \mathcal{H} \subset \mathcal{G}$ Conversely to each subalgebra $\mathcal{H} \subset \mathcal{G}$ can be associated the subalgebra $\tilde{\mathcal{H}} = \theta^{-1}(\mathcal{H}) \subset \tilde{\mathcal{G}}$ with $\tilde{\mathcal{H}} \supset \mathcal{M}$.

It follows we have the following relation between subalgebras of $\mathcal{G}$ and subalgebras of $\tilde{\mathcal{G}}$:

Subalgebras of $\mathcal{G}$	⟶	Subalgebras of $\tilde{\mathcal{G}}$:
$\mathcal{H}$		i) $\tilde{\mathcal{H}} = \theta^{-1}(\mathcal{H})$, $\tilde{\mathcal{H}} \supset \mathcal{M}$.
		ii) $\tilde{\mathcal{K}}$ subalgebras of $\tilde{\mathcal{H}}$ such that : - $\tilde{\mathcal{K}} \not\supset \mathcal{M}$ - $\dim \tilde{\mathcal{K}} = \dim \tilde{\mathcal{H}} - 1$.

It is interesting to note that the problem of determination of subalgebras of $\tilde{\mathcal{G}}$ of type ii) is simplified by the properties :

Prop. 1 : $\mathcal{H} \subset \mathcal{G}$, $[\mathcal{H}, \mathcal{H}]_{\mathcal{G}} = \mathcal{H}$ ⟹ the only subalgebra of type ii) to be considered from $\tilde{\mathcal{H}} = \theta^{-1}(\mathcal{H})$ is $\mathcal{H}$ itself.

Prop. 2 : $\mathcal{H} \subset \mathcal{G}$, $\exists A \& B \in \mathcal{H}$ s.t. $[A,B]_{\mathcal{G}} = M$ ⟹ the algebra $\tilde{\mathcal{H}} = \theta^{-1}(\mathcal{H})$ does not provide any algebra of type ii).

Moreover, the problem of classification of subalgebras of $\tilde{\mathcal{G}}$ up to a conjugation becomes easy owing to the proposition :

Proposition 3: $\mathcal{H}$ and $\mathcal{H}'$ non conjugate in $\mathcal{G}$:
⟹ α) $\tilde{\mathcal{H}} = \theta^{-1}(\mathcal{H})$ and $\tilde{\mathcal{H}}' = \theta^{-1}(\mathcal{H}')$ non conjugate in $\tilde{\mathcal{G}}$.
β) $\nexists$ $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}'$ conjugate in $\tilde{\mathcal{G}}$ with $\tilde{\mathcal{K}} \subset \tilde{\mathcal{H}}$ and $\tilde{\mathcal{K}}' \subset \tilde{\mathcal{H}}'$, $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}'$ satisfying ii).

VI - One dimensional subalgebras of $\mathcal{G}$ (and $\tilde{\mathcal{G}}$) and their characterization by invariants.

In table 3 are classified the one dimensional subalgebras of $\tilde{\mathcal{G}}$.

An analogous classification for the one-dimensional subalgebra of $\mathcal{G}$ can be easily deduced by suppressing in the above table, the families of subalgebras denoted IV and V and by taking away the condition $\gamma=0$ in I.

Representative element	Characterization by invariants
J_z $J_z + \eta H$ H	(I) $\operatorname{tg}\varphi = \dfrac{\delta}{\lvert\vec{\alpha}\rvert}$, $0 \le \varphi \le \frac{\pi}{2}$; $\begin{cases} \varphi = 0 & \vec{\alpha}\cdot\vec{\beta} = 0 \ ; \ \vec{\alpha}\cdot\vec{\gamma} = 0 \\ \varphi = \operatorname{Arctg}\eta & \vec{\alpha}\cdot\vec{\beta} = 0 \\ \varphi = \frac{\pi}{2} & \vec{\beta} = 0 \end{cases}$ $\gamma = 0$
$J_z + \beta K_z$ $J_z + \beta K_z + \delta H$ $J_z + \gamma P_z$ $K_z + \zeta H$	(II) $\operatorname{tg}\psi = \dfrac{\vec{\alpha}\cdot\vec{\beta}}{\lvert\vec{\alpha}\rvert^2}$, $0 \le \psi < \pi$; $\psi \neq \frac{\pi}{2}$; $\vec{\alpha} = 0$; $\begin{cases} \psi = \operatorname{Arctg}\beta & \begin{cases} \delta = 0 \\ \dfrac{\lvert\delta\rvert}{\lvert\vec{\alpha}\rvert} \end{cases} \\ \psi = 0 & \dfrac{\vec{\alpha}\cdot\vec{\gamma}}{\lvert\vec{\alpha}\rvert}\ \ \delta = 0 \\ & \dfrac{\lvert\delta\rvert}{\lvert\vec{\beta}\rvert} \end{cases}$
K_z $K_z + \gamma P_x$ P_z	(III) $\vec{\alpha} = 0$; $\operatorname{tg}\theta = \dfrac{\lvert\vec{\beta}\wedge\vec{\gamma}\rvert}{\lvert\vec{\beta}\rvert^2}$, $0 \le \theta \le \frac{\pi}{2}$; $\begin{cases} \theta = 0 \\ \theta = \operatorname{Arctg}\gamma \\ \theta = \frac{\pi}{2} \end{cases}$ $\delta = 0$
$J_z + \rho M$ $J_z + \eta H + \rho M$ $H + \rho M$	(IV) Same characterization as in (I) + $\begin{cases} \dfrac{\lvert(\vec{\alpha}\wedge\vec{\beta})\cdot\vec{\gamma} + \gamma\vec{\alpha}^2\rvert}{\lvert\vec{\alpha}\rvert^3} \\ \dfrac{(\vec{\alpha}\wedge\vec{\beta})\cdot\vec{\gamma} + \gamma\vec{\alpha}^2 + \frac{\delta}{2}\vec{\beta}^2 - \frac{(\vec{\alpha}\cdot\vec{\gamma})^2}{2\delta}}{\lvert\vec{\alpha}\rvert^3} \\ \dfrac{\gamma^2 - 2\gamma\delta}{\delta^2} \end{cases}$
M	(V) $\vec{\alpha} = \vec{\beta} = \vec{\gamma} = \delta = 0$

One-dimensional subalgebras of $\mathcal{G}$ ($\tilde{\mathcal{G}}$). Table 3

General element: $X = \vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K} + \vec{\gamma}\cdot\vec{P} + \delta H + \gamma M.$

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APPENDICE

[illegible]

On the automorphisms of real Lie algebras

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We establish some properties of automorphisms of real Lie algebras, which in particular allow us to construct the derivation algebra of a Lie algebra from the derivations of its radical. We apply this construction to some familiar kinematical algebras.

I. INTRODUCTION

Up to now the notion of a group of automorphisms of a finite dimensional Lie algebra has practically been used mainly in the simple case of Abelian Lie algebras for which the group of automorphisms is $Gl(n,\mathbb{R})$, where n is the dimension of the real algebra. In the non-Abelian case, two physically interesting examples are well known with the groups of automorphisms of the Poincaré[1] and the Galilei[2] algebras.

Let us add that the knowledge of the Lie algebra of automorphisms, also called the derivation algebra,[3] has been useful to classify, up to a conjugation, the subalgebras of a given Lie algebra,[4] and had led to build a theorem on the derivations of the semidirect sum of two Lie algebras $\mathfrak{N} \sqsubset \mathcal{L}$, where $\mathfrak{N}$ is Abelian and $\mathcal{L}$ semi-simple, provided that the decomposition of $\mathfrak{N}$ into invariant subspaces under $\mathcal{L}$ is unique.

In this paper it is proposed to extend the above theorem to the case of the semidirect sum $\mathcal{A} = \mathcal{R} \sqsubset \mathcal{L}$, where $\mathcal{R}$ is the radical, i.e., the maximal solvable Lie algebra in $\mathcal{A}$, and $\mathcal{L}$ plays the role of a Levi's factor. From Levi's theorem[3] any Lie algebra admits such a decomposition and it will be shown how it is possible to construct the derivations of $\mathcal{R} \sqsubset \mathcal{L}$ using some well-defined derivations of the radical, the action of these derivations being extended to the whole algebra.

To find out the group of derivations of a Lie algebra, we could have used a pure cohomological approach based on the structural theorem between cohomology groups of G. Hochschild and J.P. Serre (Theorem 13 in Ref. 5). In particular this theorem shows that the first cohomology group $H^1(\mathcal{A}, A)$, namely the quotient group of derivations by inner derivations, is essentially made of some linear maps defined in the radical into the whole algebra $\mathcal{A}$ considered as a vector space A. But these maps cannot in general be easily found. Moreover it is clear that the knowledge of the first cohomology group does not give immediately neither the structure of the derivation algebra, nor the action of this algebra on $\mathcal{A}$.

This paper is organized as follows:

Section II is devoted to some definitions and useful lemmas we need for the following sections.

The two following sections deal with some properties of $\mathcal{A}$ in connection with the automorphisms. In Sec. III it is shown that the radical $\mathcal{R}$ is stable for all automorphisms of the whole algebra $\mathcal{A}$. This result is essential and leads to the study of the derivations of $\mathcal{A}$ through the derivations of $\mathcal{R}$. The Levi's factor does not possess a so strong property. However $\mathcal{L}$ decomposes as $\mathcal{L}_{I} \oplus \mathcal{L}_{II}$ with $\mathcal{L}_{I}$ acting "effectively" on $\mathcal{R}$ and $\mathcal{L}_{I}$ commuting with $\mathcal{R}$, and it can be proved (Sec. IV) that the algebras $\mathcal{A}_{I} = \mathcal{R} \sqsubset \mathcal{L}_{I}$ and $\mathcal{L}_{II}$ are both stable for the connected part $\mathrm{Aut}^{c}(\mathcal{A})$ of the automorphisms of $\mathcal{A}$.

Section V emphasizes the role played by a particular subgroup R_0 of the group of the inner automorphisms of $\mathcal{A}_{I}$. It follows a fundamental relationship between $\mathrm{Aut}^{c}(\mathcal{A}_{I})$ and a subgroup G of $\mathrm{Aut}^{c}(\mathcal{R})$. In Sec. VI this subgroup G is fully characterized which allows to carry out the construction of the derivation algebra.

To illustrate the practical applications of this theorem, some relevant physical examples involving kinematical algebras are studied in Sec. VII.

In another paper[6] it has been shown how the notion of derivation algebra can be used to increase an explicitly time-dependent invariance algebra of a given quantum mechanical system.

II. PRELIMINARY PROPERTIES

All along this paper we are concerned with real Lie algebras.

(A) *Definition*: Let $\mathcal{A}$ be a finite dimensional Lie algebra and a an element of $\mathcal{A}$; we call ideal relative to a the subalgebra $\mathcal{J}\ (a)$ constructed as follows: let us consider the subalgebra $[\Lambda_a^1]$ obtained by closing under the Lie product the vector subspace Λ_a^1 defined as:

$$\Lambda_a^1 = \{a_0 \mid \exists\, a' \in \mathcal{A} \quad \text{such that } [a', a] = a_0\} = [\mathcal{A}, a] \quad \text{if } a \in [\mathcal{A}, a],$$

$$\Lambda_a^1 = \{[\mathcal{A}, a]\} \oplus \{a\} \quad \text{with } \{a\} = \{\lambda a \mid \lambda \in \mathbb{R}\} \quad \text{if } a \notin [\mathcal{A}, a].$$

Then let us form $[\mathcal{A}, [\Lambda_a^1]]$; two cases may appear:

(i) $[\mathcal{A}, [\Lambda_a^1]] \subset [\Lambda_a^1]$ and $[\Lambda_a^1]$ furnishes the ideal relative to a:

(ii) $[\mathcal{A}, [\Lambda_a^1]] \not\subset [\Lambda_a^1]$ then the subalgebra $[\Lambda_a^k]$ can be constructed on the vector subspace Λ_a^k deduced from the set:

$$\{[\Lambda_a^{k-1}]\} \cup \{[\mathcal{A}, [\Lambda_a^{k-1}]]\}$$

and the iterative procedure will be stopped at the first stage n for which

$$[\mathcal{A}, [\Lambda_a^n]] \subset [\Lambda_a^n],$$

then $\mathcal{J}_{\mathcal{A}}(a)$ must be identified with $[\Lambda_a^n]$.

In the case of a simple algebra the following property can be deduced.

Lemma: Let $\mathcal{S}$ be a simple algebra; $\mathcal{J}_{\mathcal{S}}(s) = \mathcal{S}$ for any

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$s \in \mathcal{S}$. Indeed the ideal $\mathcal{J}_s(s)$ being different from zero by construction, can only be, in a simple algebra, the whole algebra.

(B) *Lemma*: Let $\mathcal{L}$ be a semisimple algebra; then $[\mathcal{L}, l] \neq 0$ for any $l \in \mathcal{L}$.

The proof of this property is obvious for a simple algebra $\mathcal{S}$, since the elements $s \in \mathcal{S}$ such that $[s, \mathcal{S}] = 0$ would form an ideal in $\mathcal{S}$. We can deduce easily the same property for $\mathcal{L}$ semisimple owing to the decomposition of any semisimple algebra into a direct sum of simple subalgebras.

(C) Let $\mathcal{A}$ be a Lie algebra defined as a semidirect sum $\mathcal{R} \supset \mathcal{L}$ and $\mathcal{B}$ a proper subalgebra of $\mathcal{A}$ ($\mathcal{B} \neq 0$). We can define[4]:

$$\mathcal{L}_{\mathcal{B}} = \mathcal{L} \cap \mathcal{B} \quad \text{and} \quad \mathcal{R}_{\mathcal{B}} = \mathcal{R} \cap \mathcal{B}.$$

It can be easily seen that $\mathcal{L}_{\mathcal{B}}$ and $\mathcal{R}_{\mathcal{B}}$ are subalgebras of $\mathcal{B}$:

$$[\mathcal{L}_{\mathcal{B}}, \mathcal{L}_{\mathcal{B}}] \subseteq \mathcal{L}_{\mathcal{B}}, \quad [\mathcal{R}_{\mathcal{B}}, \mathcal{R}_{\mathcal{B}}] \subseteq \mathcal{R}_{\mathcal{B}}$$

and moreover

$$[\mathcal{L}_{\mathcal{B}}, \mathcal{R}_{\mathcal{B}}] \subseteq \mathcal{R}_{\mathcal{B}}$$

which prove that

$$\mathcal{R}_{\mathcal{B}} \supset \mathcal{L}_{\mathcal{B}} \subset \mathcal{B}.$$

If $\mathcal{R}_{\mathcal{B}} \subset \mathcal{L}_{\mathcal{B}} \subset \mathcal{B}$, we denote by $\mathcal{M}_{\mathcal{B}}$ a complementary subspace of $\mathcal{R}_{\mathcal{B}} + \mathcal{L}_{\mathcal{B}}$ in $\mathcal{B}$:

$$\mathcal{B} = \mathcal{R}_{\mathcal{B}} + \mathcal{L}_{\mathcal{B}} + \mathcal{M}_{\mathcal{B}}.$$

One can be easily convinced that any nonzero element m_j of $\mathcal{M}_{\mathcal{B}}$ can be written in an unique way as:

$$m_j = r_j + l_j,$$

with

$$r_j \in \mathcal{R}, \quad r_j \notin \mathcal{R}_{\mathcal{B}}, \quad r_j \neq 0,$$

$$l_j \in \mathcal{L}, \quad l_j \notin \mathcal{L}_{\mathcal{B}}, \quad l_j \neq 0.$$

Let us denote by $\mathcal{R}'_{\mathcal{B}}$ (respectively, $\mathcal{L}'_{\mathcal{B}}$) the subspace spanned by the r_j's (resp. l_j's), then the following property can be shown:

$$[\mathcal{L}_{\mathcal{B}} + \mathcal{L}'_{\mathcal{B}}, \mathcal{L}_{\mathcal{B}} + \mathcal{L}'_{\mathcal{B}}] \subseteq \mathcal{L}_{\mathcal{B}} + \mathcal{L}'_{\mathcal{B}},$$

i.e., that $\mathcal{L}_{\mathcal{B}} + \mathcal{L}'_{\mathcal{B}}$ is a subalgebra of $\mathcal{B}$.

(D) In the algebra $\mathcal{A}$ written as $\mathcal{R} \supset \mathcal{L}$, the Levi's factor $\mathcal{L}$ being semisimple can be decomposed into a direct sum of simple subalgebras:

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{L}_i.$$

It is useful to remark that this sum can be settled as:

$$\mathcal{L} = \left(\bigoplus_{i=1}^{k} \mathcal{L}_i \right) \oplus \left(\bigoplus_{j=k+1}^{n} \mathcal{L}_j \right) = \mathcal{L}_{\mathrm{I}} \oplus \mathcal{L}_{\mathrm{II}},$$

with $\mathcal{L}_i$, $i = 1, \ldots, k$ acting effectively on $\mathcal{R}$, i.e., such that $[\mathcal{R}_i, \mathcal{R}] \neq 0$ and $\mathcal{L}_j$, $j = k+1, \ldots, n$ commuting with $\mathcal{R}$, i.e., such that $[\mathcal{L}_j, \mathcal{R}] = 0$.

It is worth noticing the following property:

Lemma: If $\mathcal{L}$ simple acts effectively on $\mathcal{R}$, then $[l, \mathcal{R}] \neq 0$ for each $l \in \mathcal{L}$.

Proof: Let us consider the set

$$\mathcal{E} = \{ l_0 \subset \mathcal{L} \mid [l_0, \mathcal{R}] = 0 \}$$

this set is an ideal in $\mathcal{L}$ since for any $l \in \mathcal{L}$, $l_0 \subset \mathcal{E}$, the Jacobi's identity implies $[[l_0, l], r] = 0$ for each $r \in \mathcal{R}$. By hypothesis $\mathcal{E}$ is different from $\mathcal{L}$ and thus can only be zero.

QED

(E) We recall here a property given in Ref. 7.

Lemma: Let σ be a representation of a semisimple Lie algebra $\mathcal{L}$ in a finite dimensional vector space V. If W is an invariant subspace of V such that $\dim V = \dim W + 1$ and if $\sigma(l) V \subset W$ for all $l \in \mathcal{L}$, then there exists a vector $v \in V$, $v \notin W$ such that $\sigma(l) v = 0$ for all $l \in \mathcal{L}$.

III. STABILITY OF THE RADICAL

Let $\mathcal{A}$ be a real finite dimensional Lie algebra which can be written following the Levi's decomposition as the semidirect sum $\mathcal{R} \supset \mathcal{L}$ corresponding to a (nontrivial) homomorphism from $\mathcal{L}$ into $D(\mathcal{R})$ the derivation algebra of $\mathcal{R}$, where $\mathcal{R}$ is the radical, i.e., the maximal solvable ideal and $\mathcal{L}$ a Levi's factor (semisimple subalgebra by definition). Then:

Theorem 1: $\mathcal{R}$ is stable for each automorphism of $\mathcal{A}$, i.e., is a characteristic ideal of $\mathcal{A}$.

Assume that there exists an automorphism σ of $\mathcal{A}$ and an element r of $\mathcal{R}$ such that

$$\sigma(r) = l + r' \quad \text{with } l \in \mathcal{L},\ l \neq 0, \text{ and } r' \in \mathcal{R}, \tag{1}$$

$\mathcal{R}$ being an ideal of $\mathcal{A}$ $[\mathcal{A}, \mathcal{R}] \subset \mathcal{R}$ and we have $[\sigma(\mathcal{A}), \sigma(\mathcal{R})] \subset \sigma(\mathcal{R})$. But $\sigma(\mathcal{A}) = \mathcal{A}$, by definition, and it follows that

$$[\mathcal{L}, \sigma(\mathcal{R})] \subseteq \sigma(\mathcal{R}),$$

in particular,

$$[\mathcal{L}, l + r'] \subseteq \sigma(\mathcal{R}). \tag{2}$$

Obviously, $\mathcal{L}$ being semisimple can be decomposed into a direct sum of simple algebras $\mathcal{L}_i$:

$$\mathcal{L} = \bigoplus_{i=1}^{n} \mathcal{L}_i.$$

Then the element l used in (1) can be decomposed as

$$l = \sum_{i=1}^{n} l_i \quad \text{with } l_i \in \mathcal{L}_i.$$

Let us choose in this sum $l_j \in \mathcal{L}_j$, $l_j \neq 0$, which is possible since $l \neq 0$.

The relation (2) ensures that

$$[\mathcal{L}_j, l + r'] \subseteq \sigma(\mathcal{R}).$$

Let us remark that

$$[\mathcal{L}_j, l + r'] = [\mathcal{L}_j, l_j + r'].$$

Then, using the lemmas A and B, it is not difficult to deduce that $\mathcal{J}_{\mathcal{A}}(l + r')$, which is included in $\sigma(\mathcal{R})$, contains a subalgebra $\mathcal{B}$

$$\mathcal{B} = \mathcal{L}_{\mathcal{B}} + \mathcal{M}_{\mathcal{B}} + \mathcal{R}_{\mathcal{B}}$$

such that

$$L_1 + L'_1 = g_{L_1}(l_1) = L_1.$$

Therefore β is not solvable. But R solvable implies $\sigma(R)$ solvable and then $\sigma(R)$ contains only solvable subalgebras, which contradicts the above result. So an element $l \neq 0$ cannot appear in (1) and for any automorphism σ of A we have $\sigma(R) \supset R$. Since $\sigma(R)$ and R have the same dimension, it follows that R is stable under all the automorphisms of A, hence is a characteristics subalgebra of A.[1]

IV. PROPERTIES OF THE LEVI'S FACTOR

Lemma: Let π_L be the projection operator mapping A on L, considered as vector spaces, then $\pi_L \circ \sigma$ is an automorphism of L for any σ automorphism of A. In particular $\pi_L \circ \sigma$ is an inner automorphism of L if σ belongs to the connected part $\mathrm{Aut}^0(A)$ of the automorphisms of A.

Let us define

$$\sigma(l) = \lambda(l) + \rho(l), \tag{3}$$

with $\lambda(l) \subset L$ and $\rho(l) \in R$ whenever $l \in L$ and $\sigma \in \mathrm{Aut}(A)$. By definition

$$[\sigma(l_1), \sigma(l_2)] = \sigma([l_1, l_2]) \quad \text{for each pair } l_1, l_2 \in L.$$

This relation can be decomposed into the two following ones:

$$[\lambda(l_1), \lambda(l_2)] = \lambda([l_1, l_2]), \tag{4}$$

$$[\lambda(l_1), \rho(l_2)] + [\rho(l_1), \lambda(l_2)] + [\rho(l_1), \rho(l_2)] = \rho([l_1, l_2]). \tag{5}$$

Equation (4) together with the linearity property of λ ensure us that $\lambda = \pi_L \circ \sigma$ is an homomorphism of L. From Theorem 1 we can deduce that λ is a one-to-one mapping, so that λ is an automorphism of L.

In particular, if σ belongs to the connected part of the group of automorphisms of A, it is easy to show, by using the corresponding derivation, that λ is a connected automorphism of L, and thus an inner automorphism of L, since any connected automorphism of a semisimple Lie algebra is an inner one.

Let us now decompose A as $A = (R \supset L_I) \oplus L_{II}$, with L_I acting effectively on R; we have then:

Lemma 2: The algebras $R \supset L_I = A_I$ and L_{II} are stable for the connected part of the automorphisms of A. This property remains valid if $L_I = 0$.

Let us first show that $\sigma(L_{II}) = L_{II}$, for any $\sigma \in \mathrm{Aut}^0(A)$. As $[L_{II}, R] = 0$, and using also the result of the Theorem 1: $\sigma(R) = R$ we have

$$[\sigma(L_{II}), \sigma(R)] = [\sigma(L_{II}), R] = 0. \tag{6}$$

Let us consider $l_2 \in L_{II}$ and form

$$\sigma(l_2) = \lambda(l_2) + \rho(l_2).$$

Equation (6) allows us to write

$$[\lambda(l_2) + \rho(l_2), R] = 0. \tag{7}$$

Moreover we know, from Lemma 1, that if σ is connected automorphism of A, λ is an inner automorphism of L, so that[8] $\lambda(l_2) \in L_{II}$.

Then it follows from (7) that $\rho(l_2) \in C(R)$. Thus we can deduce

$$[\sigma(L_{II}), \sigma(L_{II})] \subset L_{II}. \tag{8}$$

But L_{II} being semisimple, $[L_{II}, L_{II}] = L_{II}$, and also

$$[\sigma(L_{II}), \sigma(L_{II})] = \sigma(L_{II}). \tag{9}$$

Together with (8), Eq. (9) implies $\sigma(L_{II}) = L_{II}$, since $\sigma(L_{II})$ and L_{II} have the same dimension.

Consider now the action of σ on $A_I = R \supset L_I$. We know from Theorem 1 that $\sigma(R) = R$. Moreover for any $l_1 \in L_I$ we have

$$\sigma(l_1) = \lambda(l_1) + \rho(l_1),$$

with $\lambda(l_1) \in L_I$ for the reason already mentioned.[8] Then one gets $\sigma(R \supset L_I) \subset R \supset L_I$ and $\sigma(R \supset L_I) = R \supset L_I$.

Let us mention that this proof is valid for $L_I = 0$. In this case $A = R \oplus L$, $\sigma(R) = R$, and $\sigma(L) = L$ for any $\sigma \in \mathrm{Aut}^0(A)$.

V. ROLE OF THE ALGEBRA $\mathscr{R}_0 \cong \mathscr{C}(\mathscr{R})/\mathscr{C}(\mathscr{A})$

Let R_0 be the Abelian subalgebra of the inner derivation algebra $\mathcal{ID}(A_I)$ of A_I generated from elements of the center $C(R)$ of R, namely $R_0 \cong C(R)/C(A_I)$.

Lemma 3: The Lie subgroup R_0 of the inner automorphisms of A generated by the elements of R_0 is an Abelian invariant subgroup in the group $\mathrm{Aut}(A)$ of the automorphisms of A.

Let w_{r_0} be an inner automorphism of A generated by $r_0 \in R_0$; its action on $R \supset L$ is given by

$$w_{r_0}(l) = l + [r_0, l] \quad \forall\, l \in L,$$

$$w_{r_0}(r) = r \qquad \forall\, r \in R.$$

Consider now an automorphism σ of A and form the product $\sigma^{-1} \circ w_{r_0} \circ \sigma$. One sees immediately that

$$(\sigma^{-1} \circ w_{r_0} \circ \sigma)(r) = r \quad \forall\, r \in R. \tag{10}$$

In order to study the action of this product on L we use the decomposition (3) for σ. It follows for σ^{-1}:

$$\sigma^{-1}(l) = \lambda^{-1}(l) + \rho'(l), \tag{11}$$

such that

$$\rho'(l) + \sigma^{-1}(\rho(l)) = 0 \quad \forall\, l \in L. \tag{12}$$

A simple calculation gives

$$\begin{aligned}(\sigma^{-1} \circ w_{r_0} \circ \sigma)(l) &= l + [\sigma^{-1}(r_0), l + \rho'(\lambda(l))] \\ &= l + [\sigma^{-1}(r_0), l]\end{aligned} \tag{13}$$

since $\sigma^{-1}(r_0)$ belongs to $C(R)$ [$C(R)$ characteristic subalgebra]. Let us notice that, if $\sigma^{-1}(r_0) \in C(A)$, we obtain the identity automorphism.

From Eqs. (10) and (13) we deduce that

$$\sigma^{-1} \circ w_{r_0} \circ \sigma = w_{\sigma^{-1}(r_0)} \tag{14}$$

which expresses that the w_{r_0}'s form an invariant subgroup we shall denote R_0, in $\mathrm{Aut}(A)$.

Lemma 4: (a) Every nontrivial connected automorphism of $A_I = R \supset L_I$ which acts as the identity on R is an inner automorphism w_{r_0} of A_I, generated from an element r_0 of R_0.

(b) Two connected automorphisms of $\mathcal{A}_I$, σ and σ' whose restrictions on R are identical, are R_0 equivalent, i.e., there exists an automorphism w_{r_0} generated by an element r_0 of R_0 such that

$$\sigma = \sigma' \circ w_{r_0}.$$

Let w be a connected automorphism of $\mathcal{A}_I$, $w \neq \mathbb{1}_{\mathcal{A}_I}$ such that its restriction to R is the identity $\mathbb{1}_R$ on R. For any pair $l \in \mathcal{L}_I$, $r \in R$, the relation

$$[w(l),\ w(r)] = w([l, r])$$

becomes

$$[w(l), r] = [l, r]. \tag{15}$$

As previously, we consider the decomposition

$$w(l) = \lambda(l) + \rho(l), \tag{16}$$

where $\lambda(l) \in \mathcal{L}$ and $\rho(l) \in R$.

In a first step, let us show that λ must be the identity on $\mathcal{L}_I$. Assume that $\lambda \neq \mathbb{1}_{\mathcal{L}_I}$; therefore $l_I \in \mathcal{L}_I$ exists such that

$$\lambda(l_I) - l_I \neq 0.$$

Since

$$\mathcal{L}_I = \stackrel{k}{\underset{i=1}{\oplus}} \mathcal{L}_i$$

with $\mathcal{L}_i$ simple, l_I can be decomposed as

$$l_I = \sum_{i=1}^{k} \alpha^i l_i \quad \text{with } l_i \in \mathcal{L}_i.$$

Then using Lemma 1 and Ref. 8, we can write

$$\lambda(l_I) = \sum_{i=1}^{k} \alpha^i \lambda(l_i) \quad \text{with } \lambda(l_i) \in \mathcal{L}_i.$$

In consequence

$$\lambda(l_I) - l_I = \sum_{i=1}^{k} \alpha^i (\lambda(l_i) - l_i) \neq 0,$$

ensures us that there is at least one $l_i \in \mathcal{L}_i$ ($l_i \neq 0$) such that

$$\lambda(l_i) - l_i \neq 0.$$

Equations (15) and (16) written for l_i lead to

$$[\lambda(l_i) - l_i, r] = [-\rho(l_i), r] \quad \text{for any } r \in R.$$

Owing to Lemma D of Sec. II,

$$[\lambda(l_i) - l_i, R] \neq 0.$$

Therefore $\rho(l_i) \neq 0$ and $\rho(l_i) \notin C(R)$.

Let us then consider the set

$$\mathcal{L}_0 = \{l_0 \in \mathcal{L}_i \mid \exists\, r_0 \in R^* \ [l_0, r] = [r_0, r]\ \forall\, r \in R\}.$$

It is easy to prove by repeated use of the Jacobi's identity that $\mathcal{L}_0$ is an ideal of $\mathcal{L}_i$, and that, if r_0 corresponds to $l_0 \in \mathcal{L}_0$, then $[l, r_0]$ corresponds to $[l, l_0]$ for $l \in \mathcal{L}_i$. Moreover one can verify, owing to the Lemma D, that $\mathcal{L}_0$ is isomorphic to a subalgebra of $\mathcal{L}_i$. $\mathcal{L}_i$ simple cannot contain solvable ideal other than zero: hence $\mathcal{L}_0$ is zero and λ cannot be different from $\mathbb{1}_{\mathcal{L}_I}$. Thus Eq. (15) becomes

$$[l + \rho(l), r] = [l, r] \quad \text{for any } r \in R,$$

which implies $\rho(l) \in C(R)$.

From Theorem 1 we know that the restriction of w to R, denoted w/R, belongs to $\mathrm{Aut}^c(R)$; but $C(R)$ is a characteristic subalgebra of R,[1] thus $w/C(R)$ belongs to $\mathrm{Aut}^c(C(R))$. Consequently we can consider the semidirect sum $C(R) \supset \mathcal{L}_I$ and deduce:

$$w/C(R) \subset \mathcal{L}_I \subset \mathrm{Aut}^c(C(R) \supset \mathcal{L}_I).$$

Consider now

$$w([l, l']) = [w(l), w(l')] \quad \text{for any pair } l, l' \in \mathcal{L}_I. \tag{17}$$

Using the specific form of w, Eq. (17) becomes

$$\rho([l, l']) = [l, \rho(l')] + [\rho(l), l'].$$

This relation is a 1-cocycle equation and also a 1-coboundary since $\mathcal{L}_I$ is semisimple. So that, there exists $r_w \subset C(R)$ such that

$$\rho(l) = [r_w, l], \quad l \in \mathcal{L}_I,$$

and

$$w(l) = l + [r_w, l].$$

It follows that every connected automorphism of $\mathcal{A}_I$ which reduce to the identity on R is an inner automorphism of $\mathcal{A}_I$ generated by an element of R_0.

To prove the part (b) let us first note that, if σ_1 and σ_2 are two automorphisms of $\mathcal{A}$, then:

$$\sigma_1 | R \circ \sigma_2 | R = \sigma_1 \circ \sigma_2 / R. \tag{18}$$

Consider now the two connected automorphisms σ and σ' whose restrictions on R are identical. It is easy to go back to (a) by setting

$$C = \sigma'^{-1} \circ \sigma$$

which verifies, following (18), the relation:

$$C/R = \sigma'^{-1}/R \circ \sigma/R = \mathbb{1}_R$$

Therefore there exists w_{r_0}, $r_0 \in R_0$, such that

$$\sigma'^{-1} \circ \sigma = w_{r_0}. \tag{19}$$

Since, from Lemma 3, R_0 is an invariant subgroup of $\mathrm{Aut}(\mathcal{A})$, the relation (19) defines the R_0 equivalence of σ and σ'.

From these two lemmas we deduce:

Theorem 2: The quotient group $\mathrm{Aut}^c(\mathcal{A}_I)/R_0$ is isomorphic to a subgroup G of $\mathrm{Aut}^c(R)$.

The property given in Lemma 3 allows to consider the quotient group $\mathrm{Aut}^c(\mathcal{A}_I)/R_0$ which is, according to Lemma 4, in a one-to-one correspondence φ with a subset of $\mathrm{Aut}^c(R)$. Note that φ maps any class $\bar{\sigma} \in \mathrm{Aut}^c(\mathcal{A}_I)/R_0$ on the automorphism σ/R, restriction to R of any element of $\bar{\sigma}$. Using the simple property given in Eq. (18) we can deduce that φ is an isomorphism of $\mathrm{Aut}^c(\mathcal{A}_I)/R_0$ onto a subgroup G of $\mathrm{Aut}^c(R)$.

VI. CONSTRUCTION OF THE DERIVATION ALGEBRA

Let $\mathcal{G}$ be the Lie algebra of the group G defined in Theorem 2, we have:

Theorem 3: The algebra $\mathcal{G}$ can be decomposed into two parts:

(i) The subalgebra $\hat{\mathcal{L}}_I$ which corresponds to the action of $\mathcal{L}_I$ on R in $\mathcal{A}_I$.

(ii) The subalgebra $\hat{D}$ which contains all the elements $\hat{d}$ of $D(R)$ such that $[\hat{d}, \hat{L}_I] \subset \hat{R}$, where $\hat{R}$, the algebra of the inner derivations of R ($\hat{R} \cong R/C(R)$) is an ideal in $\hat{D}$. Moreover a basis in $\hat{D}$ can be chosen such that any element of this basis is either an inner derivation, or an outer derivation $\hat{d}'$ satisfying $[\hat{d}', \hat{L}_I] = 0$. Hence $\mathcal{G}$ can be written as the semidirect sum $\mathcal{G} = \hat{D} \supset \hat{L}_I$ where $\hat{D}$ is not in general a solvable Lie algebra.

It is easily seen first, that $\hat{L}_I$ can be identified with L_I and also that the algebra $\mathcal{J}D(A_I)$ of inner derivations of A_I isomorphic to the quotient $R \supset L_I / C(R \supset L_I)$ exists in $\mathcal{G}$ modulo the Lie algebra R_0. This result can be written: $\mathcal{J}D(A_I)/R_0 \cong \hat{R} \supset \hat{L}_I$.

Consider now $\hat{d} \in \mathcal{G}$, $\hat{d}$ belonging to a complementary subspace of A_I in the vector space $\mathcal{G}$. We shall show that $[\hat{d}, L_I] \subset \hat{R}$. Indeed to the element $\hat{d}$ corresponds the class d in $D(A_I)/R_0$: $d = d + R_0$, $d \in D(A_I)$. We know that $[d, \mathcal{J}D(A_I)] \subset \mathcal{J}D(A_I)$. But, more precisely, let d_l be the element of $\mathcal{J}D(A_I)$ corresponding to the element $l \in L_I$. From the choice of $\hat{d}$ in $\mathcal{G}$, and using Lemma 1 it can be deduced that $d(l') \in R$ for any $l' \in L_I$ Then remembering that the inner derivation d_a acts on a' as:

$$d_a(a') = [a, a']$$

and using the elementary properties of the derivations:

$$[d, d_l](a') = d(d_l(a')) - d_l(d(a))$$

one deduces the action of the derivation $[d, d_l]$ on any element $a' \in A_I$, one gets

$$[d, d_l](a') = d([l, a']) - [l, d_l(a')]$$
$$= [d(l), a']$$

and since $d(l) \in R$, $[d, d_l]$ can therefore be identified with an inner derivation corresponding to an element of R. Going back to $\mathcal{G}$, we easily deduce that

$$[\hat{d}, \hat{L}_I] \subset \hat{R}. \tag{20}$$

Conversely it is possible to prove that any outer derviation $\hat{d}$ of R verifying (20) is an element of $\mathcal{G}$.

Let us show that to any such outer derivation $\hat{d}$ of R, such that $[\hat{d}, \hat{L}] \neq 0$, can be associated another outer derivation $\hat{d}'$ which differs from $\hat{d}$ by an inner derivation $\hat{d}_r$ and satisfying

$$[\hat{d}', \hat{L}_I] = 0. \tag{21}$$

Indeed, respectively, to each $\hat{d}$ defined above, one can associate the subalgebra $D_{\hat{d}}$ of the derivation algebra $D(R)$ of R, which is generated by $\hat{d}$ and all the inner derivations of $\hat{R}$. Noting that $\hat{R}$ form an ideal in $D(R)$, one can write

$$[\hat{d}, \hat{R}] \subset \hat{R}. \tag{22}$$

Equations (20) and (22) allow us to consider the following semidirect sum, subalgebra of $D(R)$: $D_{\hat{d}} \supset \hat{L}_I$. It is then interesting to notice that, from the definition of a semidirect sum, $D_{\hat{d}}$—considered as vector space—is the representation space for the representation σ of L_I associated with the semidirect sum $D_{\hat{d}} \supset L_I$. We are then in the conditions of Lemma E ($D_{\hat{d}}$ playing the role of V and $\hat{R}$ the role of W). In consequence, there exists a vector $\hat{d}' \in D_{\hat{d}}$, $\hat{d}' \notin \hat{R}$ such that

$$\hat{d}' = \hat{d} + \hat{d}_r \text{ with } \hat{d}_r \in \hat{R} \tag{23}$$

and

$$(\sigma(L_I))(\hat{d}') = [L_I, d'] = 0.$$

Now let us show that any such $\hat{d}'$ belongs to $\mathcal{G}$. Indeed, one can extend the action of $\hat{d}'$ on the whole algebra A_I by setting

$$d'(r) = \hat{d}'(r) \text{ for any } r \in R,$$
$$d'(l) = 0 \qquad \text{for any } l \in L_I. \tag{24}$$

Then, d' so defined, is a derivation of A_I; from its definition (24) we have just to verify its action on a Lie bracket $[l, r]$, where $l \in L_I$, $r \in R$,

$$d'([l, r]) = d' \circ d_l(r) = \hat{d}' \circ d_l(r) = d_l \circ d'(r) = d_l \circ d'(r),$$

and also

$$d'([l, r]) = [l, d'(r)].$$

Taking into account the action of d' on L_I, this last relation ensures that d' is a derivation of A_I.

Moreover it is easy to see that d' belongs to the class of $D(A_I)/R_0$ which corresponds to $\hat{d}'$ by the isomorphism φ defined in Sec. VII E. Therefore $\hat{d}'$ is an element of $\mathcal{G}$ and this holds for any $\hat{d}$ satisfying (20) since in the relation (23) and $\hat{d}_r$ is in $\mathcal{G}$.

The algebra $\mathcal{D}(\mathcal{A})$

It is easy now to determine $D(A_I)$ from the knowledge of $\mathcal{G}$. Indeed the subalgebra of inner derivations $\mathcal{J}D(A_I)$ is *a priori* known $\mathcal{J}D(A_I) \cong A_I / C(A_I)$ and we have just to study the Lie brackets involving outer derivations.

This can be achieved by remembering (proof of Theorem 3) that any $\hat{d}' \in \mathcal{G}$ such that $[\hat{d}', \hat{L}] = 0$ can be extended into a derivation d' on A_I, the action of which is given by Eq. (24). Thus, the action of all the elements of a basis in $D(A_I)$ is known, hence the action of the general element in $D(A_I)$ is also known, which fully characterizes the derivation algebra since, by definition, for any pair of derivations d_1 and d_2 we have

$$[d_1, d_2](a) = d_1 \circ d_2(a) - d_2 \circ d_1(a), \quad a \in A_I.$$

Finally from the property stated in Lemma 2, we deduce the whole algebra

$$D(A) = D(A_I) \oplus D(L_{II}).$$

We recall that $D(L_{II}) \approx L_{II}$, L_{II} being semisimple.

VII. APPLICATIONS TO LIE ALGEBRAS OF PHYSICAL INTEREST

In order to illustrate the above results we study the derivation algebras of some physically relevant Lie algebras. More precisely, we shall pick up examples among kinematical invariance Lie algebras which are endomorphisms of space–time or phase space.

In practice, the knowledge of the derivations of the radical R is not necessary. Indeed, $D(R)$ is a subalgebra of $\mathcal{G}l(n, \mathbb{R})$ ($n = \dim R$) and it is sufficient first to determine the generators of $\mathcal{G}l(n, \mathbb{R})$ which commute with those corresponding to the action of the semisimple

part L on R, and then to select among these elements those which are derivations of R. This can be achieved by using a $n \times n$ matricial representation of $\mathcal{G}l(n, \mathbb{R})$.

A. Algebras of the three-dimensional Euclidian group $E(3)$ and of the Pincaré group P

These two examples can be treated by the theorem of the Appendix B of the Ref. 4 already mentioned in the introduction. Indeed, these two algebras can be written $\mathcal{E}(3) = \mathfrak{A}(3) \supset SO(3)$ and $P = \mathfrak{A}(4) \supset SO(3,1)$, respectively, and $\mathfrak{A}(3)$ [respectively, $\mathfrak{A}(4)$] is an irreducible vector space under $SO(3)$ [resp. $SO(3,1)$].

To use our method, first for the Euclidean algebra $\mathcal{E}(3)$, we note that the radical $\mathfrak{A}(3)$ is Abelian and therefore its derivation algebra, which contain only outer derivations is isomorphic to $\mathcal{G}l(3, \mathbb{R})$. Hence we have to select in the algebra of $GL(3, \mathbb{R})$ the generators which commute with those of the algebra $SO(3)$ corresponding by construction to the action on $\mathfrak{A}(3)$ in the semidirect sum $\mathfrak{A}(3) \supset SO(3)$. After calculations we find there exists only one such generator which acts as a dilatation on $\mathfrak{A}(3)$. In consequence the derivation algebra can be written

$$D(\mathcal{E}(3)) = \mathfrak{A}_P(3) \supset (SO_J(3) \oplus \mathcal{R}_D).$$

In the same way, it is easy to see there exists only one generator in $\mathcal{G}l(4, \mathbb{R})$ which commutes with the generators of $SO(3,1)$ and that this generator still acts as a dilatation on $\mathfrak{A}(4)$. So we obtain the well-known result[1]

$$D(P) = \mathfrak{A}(4) \supset (SO(3.1) \oplus \mathcal{R}_D).$$

B. Algebra of the isochronous Galilei group G'

This algebra which is the derived Galilei group algebra, can be written

$$\mathcal{G}' = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)) \supset SO_J(3).$$

Although the solvable part $\mathfrak{A}$ is Abelian, the theorem of Ref. 4 cannot be applied, the decomposition of $\mathfrak{A}$ into irreducible subspaces under $SO(3)$ being not unique (more precisely, the three-dimensional vector representation appears with the multiplicity two). But from the point of view of our technic we are in the same situation as in Sec. VII A. We have to exhibit in the derivation algebra of $\mathfrak{A}$ isomorphic to $\mathcal{G}l(6, \mathbb{R})$, the maximal subalgebra of the form $\hat{D} \oplus SO_J(3)$, where $SO_J(3)$ still corresponds to the action on $\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)$ in the semidirect sum $(\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)) \supset SO_J(3)$. A simple calculation permits to show that $\hat{D}$ is isomorphic to $\mathcal{G}l(2, \mathbb{R})$. Hence the derivation algebra of $\mathcal{G}'$ is

$$D(\mathcal{G}') = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)) \supset (SO_J(3) \oplus \mathcal{G}l(2, \mathbb{R})).$$

We shall give more details about this algebra in the Sec. VII D.

C. Algebra of the Galilei group G

$$\mathcal{G} = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3) \supset \mathfrak{A}_H(1)) \supset SO_J(3).$$

The solvable part is no more Abelian. Its derivation algebra is a subalgebra of $\mathcal{G}l(7, \mathbb{R})$ and in this subalgebra there exist two generators, denoted D_1 and D_2, which commute with $SO(3)$ and which correspond to dilatations on the generators $\mathbf{P}, \mathbf{K}, H$. Thus $D(\mathcal{G})$ can be decomposed as[2]:

$$D(\mathcal{G}) = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3) \supset \mathfrak{A}_H(1)) \supset (SO_J(3) \oplus \mathcal{R}_{D_1} \oplus \mathcal{R}_{D_2})$$

and we have the following commutation relations:

$$\begin{aligned} &[D_1, H] = 2H, \quad [D_2, H] = 0, \\ &[D_1, K_j] = -K_j, \quad [D_2, K_j] = K_j, \quad (j = 1,2,3) \\ &[D_1, P_j] = P_j, \quad [D_2, P_j] = P_j, \\ &[D_1, D_2] = 0. \end{aligned} \tag{25}$$

D. Algebra of the extended isochronous Galilei group $\tilde{G}'$

To apply our method we consider this algebra written under the form

$$\tilde{\mathcal{G}}' = (\mathfrak{A}_H(1) \oplus \mathfrak{A}_P(3) \supset \mathfrak{A}_K(3)) \supset SO_J(3).$$

The solvable part is the Heisenberg algebra H of which the derivations are well known.

$$D(H) = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)) \supset (\mathrm{Sp}(6, \mathbb{R}) \oplus \mathcal{R}_D).$$

The $SO(3)$ is contained into the symplectic algebra $\mathrm{Sp}(6, \mathbb{R})$ and the generators of $D(H)$ which commute with the $SO(3)$ form a subalgebra isomorphic to $\mathcal{G}l(2, \mathbb{R})$. This algebra was already encountered in Sec. VII B where its action was limited on $\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)$. We can the write:

$$D(\tilde{\mathcal{G}}') = (\mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3)) \supset (SO_J(3) \oplus \mathcal{G}l(2, \mathbb{R})).$$

It is interesting to decompose $\mathcal{G}l(2, \mathbb{R})$ into the direct sum:

$$\mathcal{G}l(2, \mathbb{R}) \simeq SU(1,1) \oplus \mathcal{R}_{D_2}$$

which makes appear the Schrödinger algebra $\mathcal{S}$[10] defined as the largest algebra which leaves invariant the free Schrödinger equation:

$$\mathcal{S} = \mathfrak{A}_P(3) \oplus \mathfrak{A}_K(3) \supset SO_J(3) \oplus SU(1,1).$$

This algebra contains the Galilei algebra and the generator H also belongs to the $SU(1,1)$ algebra of which the two other generators are the dilatation D_1 and the "expansion" C. The new generators satisfy the following commutation relations:

$$\begin{aligned} &[C, P_j] = K_j, \quad [C, K_j] = 0, \\ &[C, J_j] = 0, \quad [C, H] = D_1, \\ &[C, D_1] = 2C, \quad [C, D_2] = 0. \end{aligned}$$

The other commutation relations are identical to the relations (25). So, we have established that the Schrödinger algebra is the derivation algebra of the extended Galilei group, up to a dilatation which acts on the mass. It may be interesting to notice that:

(i) The $SU(1,1)$ algebra which appears above is in fact the $\mathrm{Sp}(2, \mathbb{R})$ algebra which subsists in $D(H)$ if we consider only a one-dimensional space.

(ii) It is easy to convince himself that following our technics and by its construction the Schrödinger algebra is complete, i.e., it has no center and no outer derivations.

E. Algebra of the extended Galilei group $\tilde{\mathcal{G}}$

$$\tilde{\mathcal{G}} = (\mathfrak{R}_H(1) \oplus \mathfrak{R}_M(1) \oplus \mathfrak{R}_P(3) \square \mathfrak{R}_K(3)) \square SO_J(3).$$

This case differs from the case of the Galilei algebra by the appearance of a supplementary generator z, the automorphisms σ_{α_z} of which act on $\tilde{\mathcal{G}}'$ as

$$\sigma_{\alpha_z}(H) = H + \alpha M,$$

$$\sigma_{\alpha_z}(x) = x, \quad \text{for any } x \in \tilde{\mathcal{G}}'.$$

z commutes with all inner derivations, but with the two dilatations we have the following commutation relations:

$$[D_1, z] = -2z, \quad [D_2, z] = 2z.$$

Hence the derivation algebra of $\tilde{\mathcal{G}}$ can be written

$$D(\tilde{\mathcal{G}}) = (\mathcal{G}D(\mathcal{G}) \oplus R_z) \square (R_{D_1} \oplus R_{D_2}).$$

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[8] λ being an inner automorphism of $\mathcal{L} = (\oplus_{i=1}^{n} \mathcal{L}_i$ there exists $l_0 = \sum_{i=1}^{n} \alpha^i l_i$ with $l_i \in \mathcal{L}_i$ such that $\lambda(l) = \exp(l_0) l \exp(-l_0)$ for any $l \in \mathcal{L}$. In consequence, for any $l'_j \in \mathcal{L}_j$, $\lambda(l'_j) = \exp(\sum \alpha^i l_i) l'_j \times \exp(-\sum \alpha^i l_i) = \exp(\alpha^j l_j) l'_j \exp(-\alpha^j l_j)$, which proves that $\lambda(l'_j) \in \mathcal{L}_j$.
[9] Let $\mathcal{A}$ be a finite-dimensional algebra over the field of real numbers: We denote $D(\mathcal{A})$ the Lie algebra of the group of automorphisms of $\mathcal{A}$ also called the Lie algebra of derivations or derivation algebra of $\mathcal{A}$. We recall that a derivation of $\mathcal{A}$ is a linear map $d: \mathcal{A} \to \mathcal{A}$ such that

$$d[a, a'] = [d(a), a'] + [a, da'], \quad \text{for } a, a' \in \mathcal{A},$$

and that in $D(\mathcal{A})$ the Lie bracket is defined by

$$[d_1, d_2]a = d_1(d_2(a)) - d_2(d_1(a)), \quad \text{for } d_1, d_2 \in D(\mathcal{A}) \text{ and } a \in D(\mathcal{A}).$$

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